A Proposal of a Shape-Optimization Method Using a Constitutive Equation of Growth*
(In the Case of a Static Elastic Body)

Hideyuki AZEGAMI**

A simple method for analysis of uniform-strength shape is newly proposed. In this paper, the most fundamental case of a static elastic body is considered. The idea of the present method came from the growth behavior of living organisms by which they changed their own shapes to adapt themselves to the mechanical living environment. The scheme consists of the iteration of the two analytical steps: (1) conventional elastic analysis for evaluation of stress distribution, and (2) incremental growth analysis using a constitutive equation of growth. In the latter step, a shape is deformed with an incremental bulk strain which is generated according to an objective stress indicating strength of the material. Two examples of a cantilever beam under top shear loading and a column under top compressive loading and gravity are analyzed to show the effectiveness of the proposed method.

Key Words: Optimum Design, Computational Mechanics, Numerical Analysis, Finite-Element Method, Constitutive Equation, Uniform Strength

1. Introduction

The objective of the present paper is to propose a new approach to the analysis of uniform-strength shapes for general use. In this paper, the most fundamental case of a static elastic body is considered.

The motive of the present work was a hypothesis described in the explanation by Fung and Seguchi[1]: "In biosystems, the behavior of growth and atrophy is most likely controlled by a stress, and the change in the residual stress may be caused by the behavior (in Japanese)." I believed that if we could implement an incremental growth analysis with the finite-element method obeying a constitutive equation of growth (or growth law) in which swelling and contracting bulk strains sprang up in response to stress, shapes could be deformed and optimized like biosystems.

The present method is positioned vis-à-vis the previous methods as following. In the various classifications, one is to assort the methods into two groups in terms of the definition of optimum criteria: (1) "minimum criterion" of an object function and (2) "uniform criterion" of a distributed object function, where the term uniformity is defined as a stationary state of an iterative uniformalizing process. The minimum condition of the potential energy, which means the minimum deflection at loading points, under volume constant is an instance of the former. For these problems, the previous numerical methods were generally based on mathematical programming[2]~[5]. The uniform condition of the equivalent stress under Mises's criterion to strength of material is an instance of the latter. The previously proposed numerical methods were to transform the finite elements adjoining a surface with a particular pattern[6][7] or to shift the nodal positions on a surface according to a magnitude of a reference stress[8]. The proposed method is classified into the latter, but differs from these previous approaches in the use of the bulk strain which is generated on the constitutive equation of growth in all parts of the body.

Simplicity of the procedure is a feature of the proposed method, particularly in comparison with the

* Received 10th July, 1989. Paper No. 88-0125A
** Department of Energy Engineering, Toyohashi University of Technology, 1-1 Hibarigaoka, Tempaku-cho, Toyohashi, 440, Japan

mathematical programming approach. Indeed, the mathematical programming approach required calculation of the sensitivity of the object function to design variable of shape, so that steep extension of computer memory and additional programming for optimization were required. In contrast, the proposed approach does not require calculation of the sensitivity because of the use of the initial- stress method[9], which is popular for nonelastic analyses, so that the programming is simplified and steep extension of computer memory is unnecessary.

The present paper consists of the following parts: section 2 ('Growth Law and Shape Optimization') gives the formulation of the constitutive equation of growth and proposes the scheme for shape optimization, section 3 ('Numerical Method with Finite Elements') presents the method of analysis, and section 4 ('Examples') shows the fundamental experience of computation.

2. Growth Law and Shape Optimization

In this section, we formulate the constitutive equation of growth which makes strength uniform when a criterion of strength is given to a fracture or an elastic failure of material, and propose the scheme for shape optimization.

The criterion of strength is generally given with a measure of a specific stress. In the case of the maximum principle-stress criterion, which has been known as an effectual criterion to the fracture of brittle materials, the maximum principle stress is employed as the measure. The equivalent stress is used in Mises's criterion to the elastic failure of ductile material. The former criterion is given by the inequality

$$\sigma_1 \geqq \sigma_c, \tag{1}$$

where σ_1 is the largest principle stress, and the latter is given with the equivalent stress σ_{eq}

$$\sigma_{eq}=\left[\frac{1}{2}(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+(\sigma_{11}-\sigma_{22})^2 +6(\sigma_{23}^2+\sigma_{13}^2+\sigma_{12}^2)\right]^{1/2} \tag{2}$$

by the inequality

$$\sigma_{eq} \geqq \sigma_c, \tag{3}$$

where σ_c is a critical stress of the fracture or the elastic failure, and σ_{ij} is the stress tensor.

Therefore the objective for our shape optimization is to uniformalize a given stress from these stresses. Then we represent the stress as a object stress σ_{obj}. which is found as a distributed object function described in the introduction.

Let the uniformalized condition be the stationary state of an iterative uniformalizing process, as mentioned in the introduction, and the uniformalizing process be an incremental growth process with the following law. We formulate the growth law in which a swelling bulk strain springs up in the case of the object stress over its basic value, and a contracting bulk strain springs up in the opposite case.

Concretely, in the elastic analysis, the constitutive equation is given by Hook's law:

$$\sigma_{ij}=D_{ijkl}\varepsilon_{kl}^E, \tag{4}$$

where σ_{ij}, ε_{kl}^E, and D_{ijkl} are the stress tensor, the elastic strain tensor, and the elastic stiffness tensor. In this paper, the summation convention is employed in tensor expression. In the incremental growth process, we assume that the incremental total strain $\Delta\varepsilon_{kl}$ consists of the incremental elastic strain $\Delta\varepsilon_{kl}^E$ and the incremental bulk strain $\Delta\varepsilon_{kl}^B$, and the relation by Hook's law holds.

$$\Delta\sigma_{ij}=D_{ijkl}\Delta\varepsilon_{kl}^E=D_{ijkl}(\Delta\varepsilon_{kl}-\Delta\varepsilon_{kl}^B). \tag{5}$$

Where the incremental bulk strain $\Delta\varepsilon_{kl}^B$ is generated in response to the deflection of the object stress σ_{obj} to the basic stress σ_{bas}:

$$\Delta\varepsilon_{kl}^B=\Delta f(\sigma_{obj}-\sigma_{bas})\delta_{kl}. \tag{6}$$

The tensor δ_{kl} denotes the Kronecker delta, and $\Delta f(\sigma_{obj}-\sigma_{bas})$ represents an incremental growth function. A straightforward incremental growth function is given by the proportional relation:

$$\Delta f(\sigma_{obj}-\sigma_{bas})=\frac{\sigma_{obj}-\sigma_{bas}}{\sigma_{bas}}\Delta h, \tag{7}$$

where Δh is a constant with which the magnitude of the incremental bulk strain is determined at an iteration, so we term it an incremental growth rate. The basic stress σ_{bas} might be regarded as a design constant or an average in volume; it should be considered as a variable when we set a ceiling on the maximum of the object stress.

A schematic flow chart for shape refinement is shown in Fig. 1. In this chart, the control of the basic stress is omitted to maintain the simplicity. The method starts at the input of an original shape (1). Based on the shape, the conventional elastic analysis (2) is followed and from it, the distribution of object stress is output. The boundary condition is set up from a mechanical condition of the problem. The convergence of the the object stress distribution (3) is judged after the elastic analysis. At the judgement, when the convergence is confirmed, the shape is output (4); when it is not confirmed, the incremental growth analysis (5) is followed. On the incremental growth analysis, the object stress distribution is input and the incremental growth displacement is output. The boundary condition is given from a restriction of shape deformation for design. The shape modification (6) is performed by moving the nodes based on the incremental growth displacement. After the modification, the flow returns to the elastic analysis (2). The indicator of the convergence might set the object stress ratio of the maximum value to the basic stress. In the case in which a finite-element mesh is distorted by the growth deformation, an implementation to regularize the mesh is required in the shape modification.

3. Numerical Method with Finite Elements

In this section, the numerical method of the two analytical steps with finite elements is presented. The substance of the method for the incremental growth analysis is the initial stress method[9], which is well known as a general method for a nonelatic deformation analysis. The vector and matrix expression is used in this section.

Based on the standard finite-element procedure,

$$\{u(\boldsymbol{x})\}=[N(\boldsymbol{x})]\{u\} \tag{8}$$

$$\{\varepsilon(\boldsymbol{x})\}=[B(\boldsymbol{x})]\{u\} \tag{9}$$

where $\{u\}$ is the nodal displacement vector, $\{u(\boldsymbol{x})\}$ and $\{\varepsilon(\boldsymbol{x})\}$ are the inner displacement and the inner strain vectors, and $[N(\boldsymbol{x})]$ and $[B(\boldsymbol{x})]$ are the shape function and the nodal displacement-strain matrices. The bolt sign $\boldsymbol{x}$ denotes the coordinate vector in a finite element and it will not be omitted for its functions in this paper.

The constitutive equation of elasticity is given by Eq. (4):

$$\{\sigma(\boldsymbol{x})\}=[D]\{\varepsilon^E(\boldsymbol{x})\}=[D][B(\boldsymbol{x})]\{u\}. \tag{10}$$

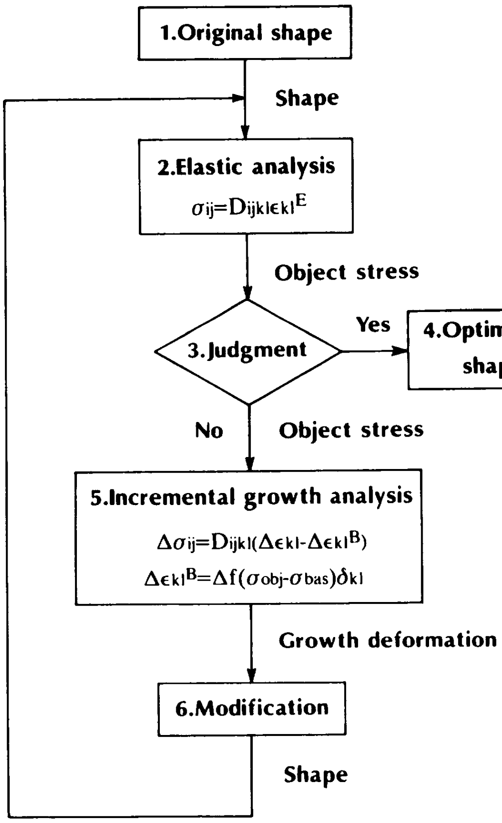


Fig. 1 Shape-optimization method with growth law

Then substitution of Eqs. from (8) to (10) into the principle of virtual work

$$\int_{V^e}\delta\{\varepsilon^E(\boldsymbol{x})\}^T\{\sigma(\boldsymbol{x})\}dV-\int_{V^e}\delta\{u(\boldsymbol{x})\}^T\{b(\boldsymbol{x})\}dV$$
$$-\int_{S_\sigma^e}\delta\{u(\boldsymbol{x})\}^T\{t(\boldsymbol{x})\}dS=0, \tag{11}$$

gives the element-governing equation for the elastic analysis:

$$[k]\{u\}=\{b\}+\{t\}, \tag{12}$$

where

$$[k]=\int_{V^e}[B(\boldsymbol{x})]^T[D][B(\boldsymbol{x})]dV \tag{13}$$

$$\{b\}=\int_{V^e}[N(\boldsymbol{x})]^T[b(\boldsymbol{x})]dV \tag{14}$$

$$\{t\}=\int_{S_\sigma^e}[N(\boldsymbol{x})]^T\{t(\boldsymbol{x})\}dS. \tag{15}$$

The vectors $\{b(\boldsymbol{x})\}$ and $\{t(\boldsymbol{x})\}$ are the body force and the traction; V^e and S_σ^e are the element volume and the surface on the element over which the traction force is prescribed; $[k]$, $\{b\}$ and $\{t\}$ are the element stiffness matrix, the equivalent nodal force vectors by the body force, and the traction; and the sign denotes the virtual variation subjected to the boundary condition of displacement. The global governing equation is given by superimposing every element-governing equation.

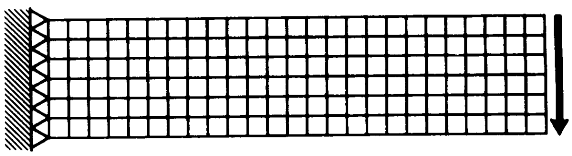
The constitutive equation for incremental growth is given by Eq. (5):

$$\{\Delta\sigma(\boldsymbol{x})\}=[D]\{\Delta\varepsilon(\boldsymbol{x})\}-[D]\{\Delta\varepsilon^B(\boldsymbol{x})\}$$
$$=[D][B(\boldsymbol{x})]\{\Delta u\}-[D]\{\Delta\varepsilon^B(\boldsymbol{x})\}. \tag{16}$$

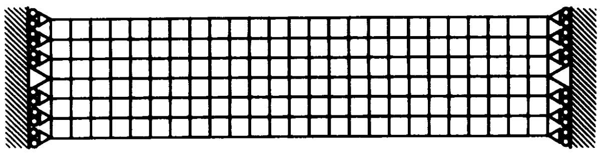
Since the external forces do not change during the incremental growth, substitution of the incremental relations of Eqs. (8) and (9) and Eq. (16) into the incremental relation of Eq. (11)

$$\int_{V^e}\delta\{\Delta\varepsilon(\boldsymbol{x})\}^T\{\Delta\sigma(\boldsymbol{x})\}dV=0, \tag{17}$$

gives the element-governing equation for the in-

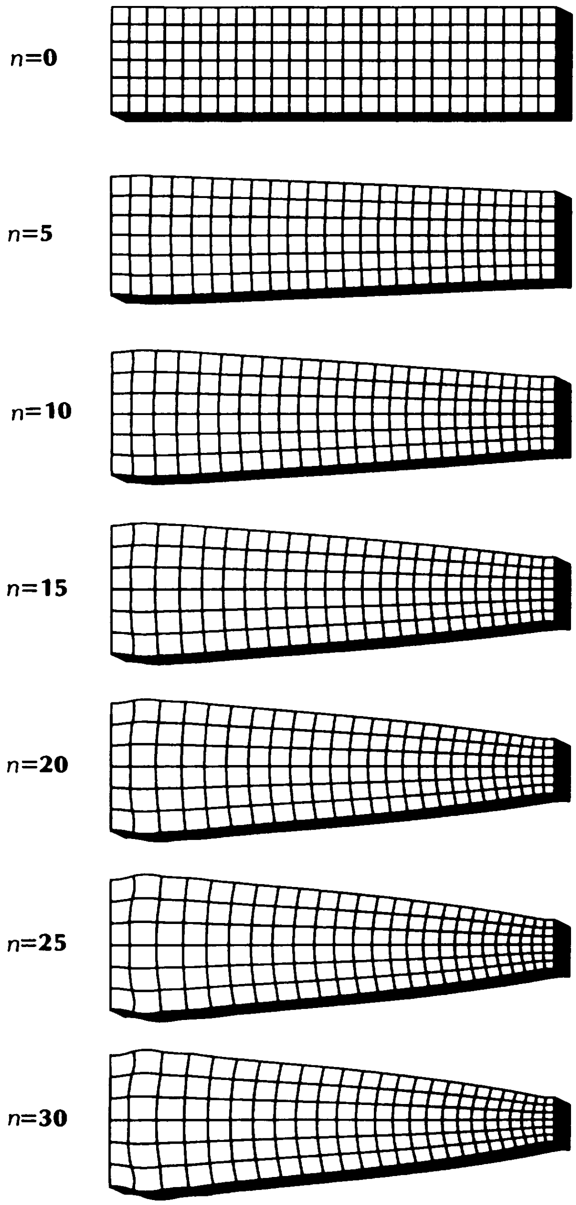


(a) for elastic analysis

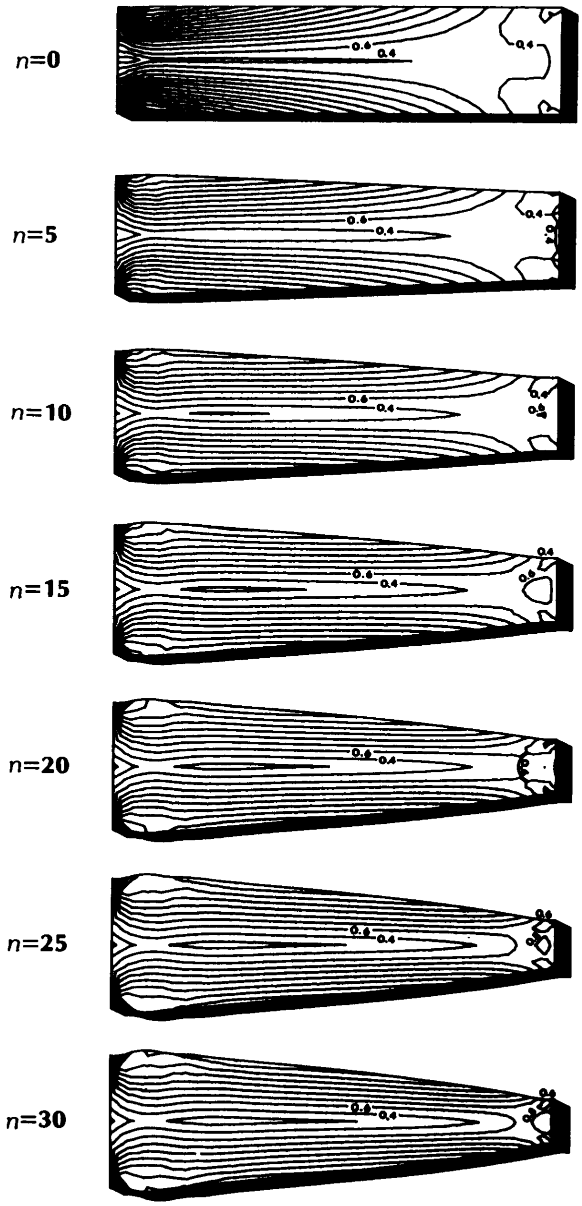


(b) for incremental growth analysis

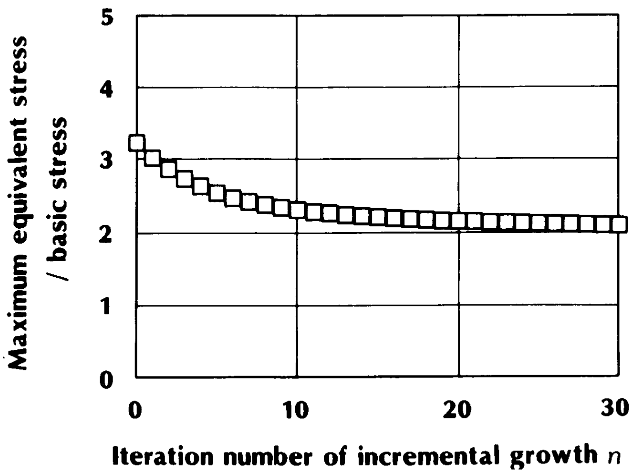
Fig. 2 Cantilever beam under top shear loading: Boundary conditions; width = 1.2 m, length = 5 m; Young's modulus = 210 GPa, Poisson's ratio = 0.3 and load = 6 MN/m.



(a) meshes



(b) contours of equivalent stress ratio to basic stress



(c) convergence rate

Fig. 3 Results of the shape-optimization process in the cantilever beam problem.

cremental growth analysis:

$$[k]\{\Delta u\}=\{\Delta g\}, \tag{18}$$

where

$$[k]=\int_{V_e}[B(\boldsymbol{x})]^T[D][B(\boldsymbol{x})]dV, \tag{19}$$

$$\{\Delta g\}=\int_{V_e}[B(\boldsymbol{x})]^T[D]\{\Delta\varepsilon^B(\boldsymbol{x})\}dV. \tag{20}$$

The matrix $[k]$ is the same as the stiffness matrix of Eq. (13), and $\{\Delta g\}$ is an equivalent nodal force vector generated by the bulk strain. The global governing equation is given by superimposing every element-governing equation.

Therefore the numerical procedure is obtained in the following order. The elastic nodal displacement, as a matter of course, is obtained by solving the global governing equation of elasticity. The distribution of object stress is evaluated with the stress which is calculated by Eq. (10). With the object stress distribution, the distribution of incremental bulk strain is given by Eqs. (6) and (7). The nodal displacement of incremental growth is obtained by solving the global governing equation for the incremental growth.

4. Examples

Two examples of plane-strain problems were provided to confirm the propriety of the proposed method.

For the examination, a program using the eight-noded isoparametric element was prepared. The object stress was assumed to be the equivalent stress in Mises's criterion. For the incremental growth function, we used Eq. (7), in which the value of 0.05 was taken for the incremental growth rate Δh. As an indicator of the convergence, the object stress ratio of the maximum value to the basic stress was evaluated, where the basic stress was given with the average of the object stress distribution in volume and the maximum value was estimated in the values at the four points to every element of Gaussian integral points. For mesh refinement, the inner nodes were rearranged at regular intervals, in the direction of width in both examples. The prepared programme is equipped with a function of drawing contours of the object stress. The methods to remove the discontinuity of the object stress distribution between elements and to draw the contours are described in the appendix. The examination was made on a microcomputer (NEC PC-9801VM2 with numeric data processor 8087).

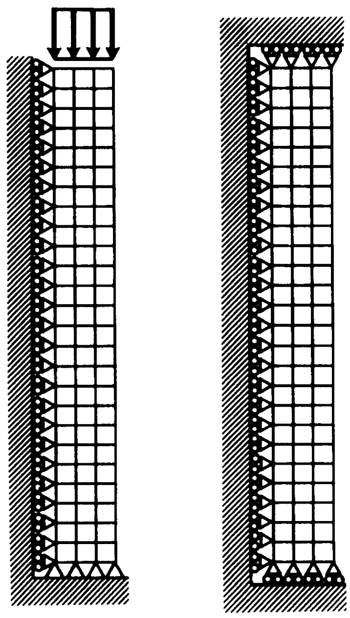
4.1 Cantilever beam under top shear loading

The prepared conditions are shown in Fig. 2. The shear loading was assumed to be distributed uniformly at the top and did not change its total value of 6 MN/m. The grid represents the finite-element meshes. Figure 3 shows the results of the shape optimization process to the iteration of the incremental growth.

The results show that the present approach was successful on this problem, since the distribution of the object stress was uniformalized by the iteration.

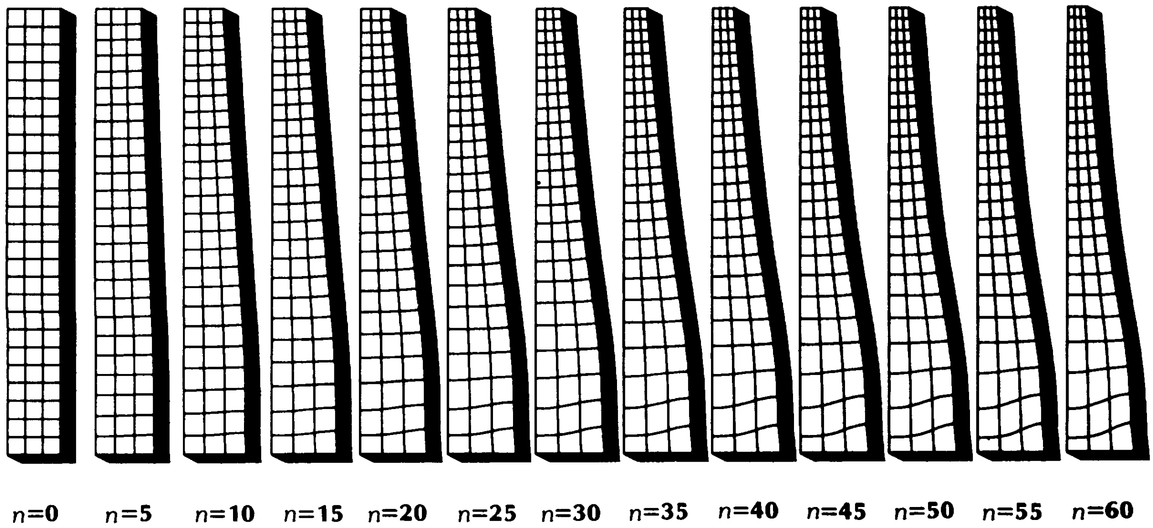
As might be expected, the refined shape was similar to the Euler beam or the Timmoshenko beam with uniform strength except in the vicinity of the fixed line. In the former beam, the shear direction width was to be a parabolic state, and in the latter beam, because the shear stress was taken into account, the parabolic state was corrected by becoming a finite width near the top. The difference in the vicinity of the fixed bottom can be seen as an effect of continuum with finite width. Indeed we can see that the equivalent stress on the fixed line was smaller than that slightly apart from the fixed line, since the equivalent stress was a function of the deviatoric stress and the hydrostatic pressure was higher and the deviatoric stress was lower on the fixed line than those slightly apart from the fixed line.

Moreover, the tendency for the object stress ratio of the maximum value to the basic stress to decrease with the iteration of incremental growth in an exponential form was expected from Eq. (7), because the incremental bulk strain sprang up in proportion with the deflection of the object stress to the basic stress, and the difference between the object

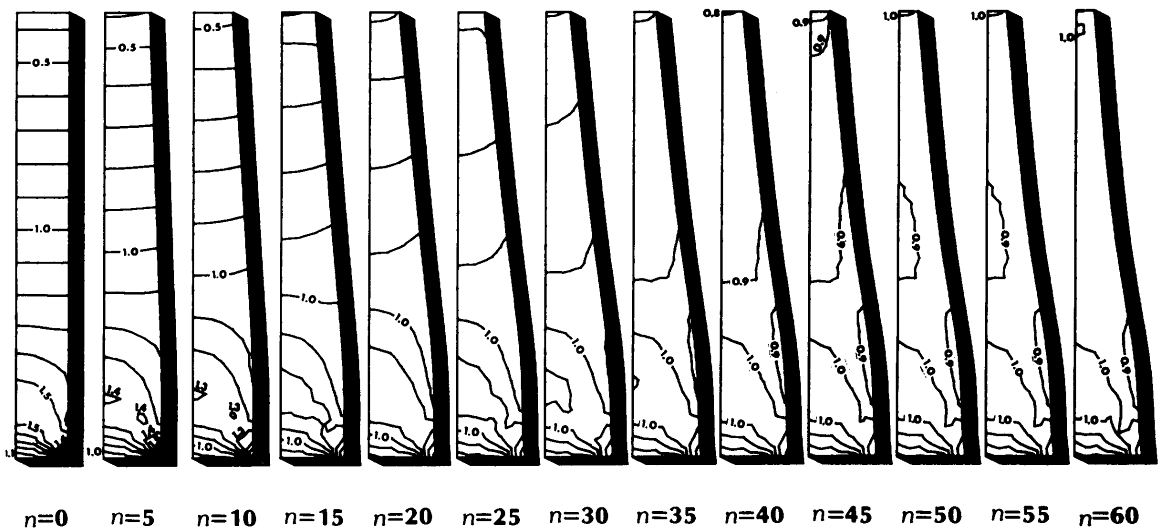


(a) for elastic analysis (b) for incremental growth analysis

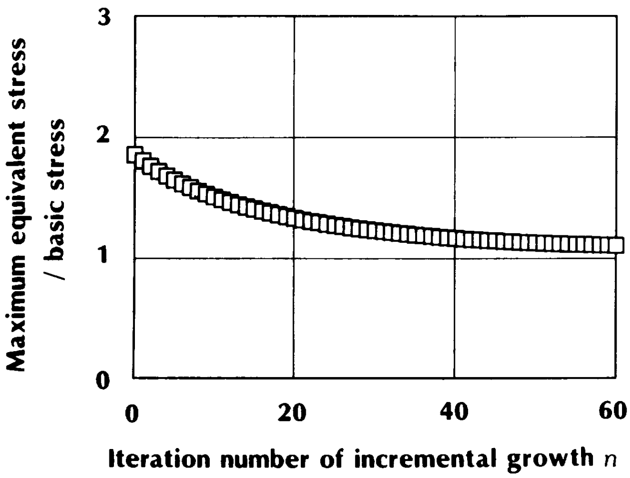
Fig. 4 Column under top compressive loading and gravity: Boundary conditions ; width/2 = 0.6 m, length = 5 m ; Young's modulus = 210 GPa, Poisson's ratio = 0.3, density = 7.86×10^3 kg/m^3 and load/2 = 0.06 MN/m.



(a) meshes



(b) contours of equivalent stress ratio to basic stress



(c) convergence rate

Fig. 5 Results of the shape-optimization process in the column problem.

stress and the basic stress was decreased almost in proportion with the shape modification rate, which was determined by the magnitude of the bulk strain.

In order to confirm the convergence state, an additional examination with a smaller incremental growth rate of 0.01 was performed following the converged result. A new convergence state, however, was not found.

4.2 Column under top compressive loading and gravity

The defined conditions are shown in Fig. 4. Due to symmetry, half of the column was analyzed. External forces were the gravity (gravitational acceleration: 9.8 m/s^2) and it was assumed that the compressive loading distributed uniformly at the top and did not change its total value of 0.06 MN/m in a half of the column. Figure 5 shows the results of the shape optimization process.

The results demonstrate that the present approach was also successful on this problem, since the distribution of the object stress was uniformalized by the iteration of the incremental growth.

As might be expected, the refined shape was similar to the simple column with uniform strength based on the assumption of uniform stress in cross section or of the infinitesimal area of the cross section except in the vicinity of the fixed bottom. In the simple column the cross-sectional area changed in the form of exponential function. The difference in the vicinity of the fixed bottom can also be seen as an effect of continuum with finite cross-sectional area. Indeed, we can see that the weight was suspended on the center rather than on both sides of the bottom.

5. Conclusions

Based on an idea which was suggested from the growth behavior of biosystems, a simple shape-analysis method for uniformalizing strength was proposed. The scheme is shown in Fig. 1. The shape deformation was performed with bulk strain which was generated according to the object stress indicating strength in the incremental growth step. The finite-element method was employed for the numerical analyses and the initial stress method was employed for the incremental growth analysis.

The two examinations of the cantilever beam under top shear loading (Fig. 2) and the column under top compressive loading and gravity (Fig. 4) indicated the effectiveness of the proposed method for uniformalizing strength.

6. Acknowledgements

This study was conducted with the support of Professor Akiyoshi OKITSU, Toyohashi University of Technology, and the cooperation of Akiyasu TAKAMI, undergraduate student, Toyohashi University of Technology. The author would like to thank them.

7. Appendix: Drawing of Object Stress Contours

In the typical elastic analysis, the continuity of the displacement is held between elements, but that of the strain or the stress is not held. While the most exact expression of the object stress contours is to draw them independently at every element with the discontinuity, this expression might be difficult to read. Therefore in the prepared program, the functions to smooth the discontinuous distribution of the object stress and to draw the contours were provided.

We assumed that the smoothed object-stress distribution was given with the shape function $\{N(\boldsymbol{x})\}$:

$$\sigma_{objs}(\boldsymbol{x})=\{N(\boldsymbol{x})\}^T\{\sigma_{objs}\}, \quad (21)$$

where $\{\sigma_{objs}\}$ and $\sigma_{objs}(\boldsymbol{x})$ are the nodal vector and the inner value of smoothed object stress in an element,

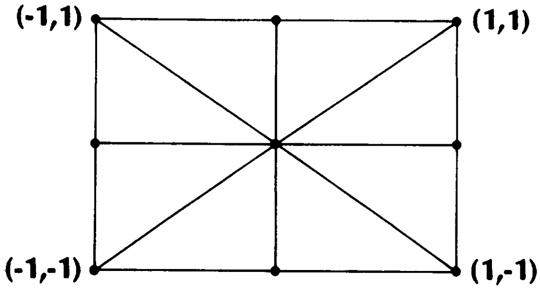


Fig. 6 Division of an eight-noded isoparametric element into eight triangles for drawing of object stress contours.

respectively. We assumed the form:

$$\int_{V_e}\{N(\boldsymbol{x})\}(\sigma_{objs}(\boldsymbol{x})-\sigma_{obj}(\boldsymbol{x}))dV=\{0\}, \quad (22)$$

where $\sigma_{obj}(\boldsymbol{x})$ is the original inner object stress with the discontinuity. Then substitution of Eq. (21) into Eq. (22) gives the equation:

$$(1/\rho)[m]\{\sigma_{objs}\}=\{\sigma_{obj}\}, \quad (23)$$

where

$$(1/\rho)[m]\{\sigma_{objs}\}=\int_{V_e}\{N(\boldsymbol{x})\}\{N(\boldsymbol{x})\}^T dV, \quad (24)$$

$$\{\sigma_{obj}\}=\int_{V_e}\{N(\boldsymbol{x})\}\sigma_{obj}(\boldsymbol{x})dV. \quad (25)$$

Here $[m]$ is termed the element mass matrix when ρ is to be the density. The global equation was given by superimposing every element equation. Therefore the nodal vector of the smoothed object stress was obtained by solving the global equation and the inner distribution of the smoothed object stress was given by Eq. (21).

The contours of the smoothed object-stress distribution were drawn in the eight triangles in every element, as shown in Fig. 6, with straight lines, since the eight nodal values of the smoothed object stress were already obtained, and the center value on the normalized coordinates was given by Eq. (21).

References

(1) Fung, Y. C. and Seguchi, Y., Mechanics Applied to Living Systems, Jour. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 88, No. 796 (1985-3), p. 290.

(2) Zienkiewitz, O. C. and Campbell, J. S., Shape Optimization and Sequential Linear Programming, Gallagher, R. H. and Zienkiewicz, eds., O. C., Optimum Structural Design (Theory and Applications), (1973), John Wiley & Sons, p. 109.

(3) Seguchi, Y. and Tada, Y., Shape Determination Problems of Structures by Inverse Variational Principle (The Finite Element Formulation), Acta Tech. ČSAV, Vol. 24-2 (1979), p. 139.

(4) Ōkouchi, T. and Miyata, S., Optimal Shape Design of Mechanical Structure (1 st Report, Discreted Shape Transformation), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 49, No. 448, A (1983-10),

p. 1532.
(5) Okouchi, T. , Torii, M., Sakai H. and Fukushima, M., Optimum Shape Design of Mechanical Structure (2nd Report, Inquiry of Sensitivity), Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 52, No. 474, A (1986-2), p. 377.
(6) Oda, J., On a Technique to Obtain an Optimum Strength Shape by the Finite-Element Method, Bull. JSME, Vol. 20, (1977), p. 160.
(7) Oda, J. and Yamazaki, K., On a Technique to obtain an Optimum Strength Shape by the Finite-Element Method: (Application to the Problem under Body Force, Bull. JSME. Vol. 22, (1979), p. 131.
(8) Umetani, Y. and Hirai, S., An Adaptive Shape Optimization Method for Structural Material Using the Growing-Reforming Procedure, 1975 Joint JSME-ASME Appl. Mech. Western Conf., Honolulu, Hawaii, (1975), p. 359.
(9) Zienkiewicz, O. C., The Finite-Element Method, 3rd Ed., (1977), McGraw-Hill.