

Domain Optimization Analysis in Linear Elastic Problems*

(Approach Using Traction Method)

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We present a numerical analysis and results using the traction method for optimizing domains in terms of which linear elastic problems are defined. In this paper we consider the application of the traction method, which was proposed as a solution to domain optimization problems in elliptic boundary value problems. The minimization of the mean compliance is considered. Using the Lagrange multiplier method, we obtain the shape gradient functions for these domain optimization problems from the optimality criteria. In this process we consider variations in the surface force acting on the boundary and variations in the stiffness function and the body force distributed in the domain. We obtain solutions for an infinite plate with a hole and a rectangular plate clamped at both ends.

Key Words: Optimum Design, Computer-Aided Design, Numerical Analysis, Computational Mechanics, Finite-Element Method, Domain Optimization, Linear Elastic Problem, Gradient Method, Traction Method

1. Introduction

Domain optimization problems in linear elastic continua assuming a geometrical domain shape as a design variable are common problems that arise in the design process for solid structures. In this paper, we propose a numerical analysis method applicable to these problems.

In order to optimize the domains, one practical approach is to provide a substitute model with a finite number of degrees of freedom as a continuum model before formulation of the optimization problems. Based on this model, a shape optimization problem, in which the design variables are defined in a finite-dimensional vector space, can be analyzed numerically using the mathematical programming techniques

in the same way as common size optimization problems. A method in which design variables are defined by nodal coordinates on a design boundary in a finite element model has been examined since the early 1970s⁽¹⁾. Using this approach, the oscillation of the design boundary is observed during the optimization process. Methods to overcome this deficiency are based on the adaptive finite element method⁽²⁾. An alternate way of defining design variables is to use the degrees of freedom of the B-spline functions representing the design boundary⁽³⁾. This method, however, is not effective in locating an optimal solution in problems with a large number of design variables because of the large number of dimensions in the design space.

An alternate approach to the optimization of geometrical domains is to describe the problem using a distributed mapping function, the derivative of which with respect to shape variation corresponds to a velocity field. The governing equation of the velocity field derived by applying the gradient method for distributed parameter systems is then solved. Using this approach, a sensitivity function, which we call a shape gradient function, can be derived theoretically as a coefficient function of the velocity field. A

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numerical analysis method is formulated to solve the governing equation. We proposed a traction method for solving domain optimization problems in which elliptic boundary value problems are defined. In this method, the velocity fields are obtained as solutions of pseudoelastic problems in pseudolinear elastic continua defined on the design domain and loaded with pseudodistributed external force, or traction, in proportion to the shape gradient function in the design domain under constraints on displacement of the invariable boundaries. This solution is called the traction method. The pseudolinear elastic problem can be analyzed using any numerical analysis technique applicable to linear elastic problems, such as the finite element method or boundary element method.

In this paper, we apply the traction method to the optimization of domains in linear elastic continua. Considering the mean compliance minimization problem, we present the following: (1) the theoretical derivation of the optimum criteria and the shape gradient function assuming that a boundary loaded with an external boundary force can be varied and that the body force and elastic stiffness can have nonuniform distributions; (2) the validity of the numerical analysis method based on the traction method using the derived shape gradient function. The notation used in the linear elastic problem and domain variation is presented first. Using this notation, we derive the mean compliance minimization formula and deduce the optimum criteria and shape gradient functions. Then the traction method is introduced to solve these problems. Finally a discussion based on the numerical examination is presented.

2. Linear Elastic Problem

Let us define the notation used in the linear elastic problem.

A linear elastic continuum is defined in an open domain $\Omega \subset R^n$, $n=2, 3$, with a boundary Γ . We consider that a coercive displacement \mathbf{h} distributed on $\Gamma_1 \subset \Gamma$, a volume force \mathbf{f} in Ω and a traction \mathbf{P} on the

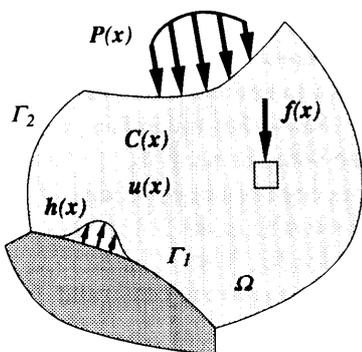


Fig. 1 Linear elastic problem

boundary $\Gamma_2 = \Gamma \setminus \Gamma_1$ yield a displacement \mathbf{u} in Ω . R is the set of real numbers and \setminus indicates the subtraction between sets.

For the elastic continuum, the variational form, or the weak form, of the equilibrium equation is expressed as

$$a(\mathbf{v}, \mathbf{w}) = l(\mathbf{w}) - l_h(\mathbf{w})$$

$$\mathbf{v} = \mathbf{u} - \mathbf{h} \in H_{\Gamma_1}, \quad \forall \mathbf{w} \in H_{\Gamma_1}, \tag{1}$$

where the bilinear form $a(\cdot, \cdot)$ and the linear forms $l(\cdot)$ and $l_h(\cdot)$ are defined by

$$a(\mathbf{v}, \mathbf{w}) = \int_{\Omega} C_{ijkl} v_{k,l} w_{i,j} dx \tag{2}$$

$$l(\mathbf{w}) = \int_{\Omega} f_i w_i dx + \int_{\Gamma_2} P_i w_i d\Gamma \tag{3}$$

$$l_h(\mathbf{w}) = \int_{\Omega} C_{ijkl} h_{k,l} w_{i,j} dx, \tag{4}$$

and the kinetically admissible set of displacements by $H_{\Gamma_1} = \{\mathbf{v} \in H^1(\Omega) | \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}$.

The Hooke elasticity \mathbf{C} , $C_{ijkl} \in (L^\infty(\Omega))^{n^4}$, has symmetry and positive definiteness given by

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{6}$$

$$\exists \alpha > 0 : C_{ijkl} \xi_{kl} \xi_{ij} \geq \alpha \xi_{ij} \xi_{ij}, \quad \xi_{ij} = \xi_{ji}$$

$$\forall \xi \in R^{n^2} \text{ in } \Omega. \tag{7}$$

The functions defined above are assumed to be $\mathbf{f} \in H^{-1}(\Omega)$, $\mathbf{h} \in H^1(\Omega)$ and $\mathbf{P} \in H^{-1/2}(\Gamma_2)$.

In this paper, boldface vector notation and tensor notation with subscripts are used. In the tensor notation, the Einstein summation convention and gradient notation $(\cdot)_{,i} = \partial(\cdot)/\partial x_i$ are used. $L^\infty(\Omega)$ and $H^m(\Omega)$ denote the bounded Lebesgue functional space and the Sobolev space, respectively.

3. Domain Variation

The domain variation can be denoted by the material derivative method^{(4),(5)} as presented in the previous paper⁽⁶⁾.

Let a domain Ω be variable in an admissible domain D with a partially smooth boundary as shown in Fig. 2. The change of domain Ω to domain Ω_s can

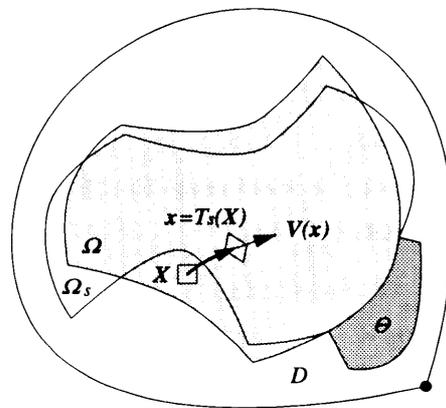


Fig. 2 Domain variation

be described using a one-to-one mapping $T_s(\mathbf{X})$ defined in the closed domain \bar{D} by

$$T_s(\mathbf{X}) : \bar{D} \ni \mathbf{X} \mapsto \mathbf{x}(s) \in \bar{D}, \quad (8)$$

where nonsmooth points on the boundary of D , if they exist, are fixed to avoid changing the admissible domain D . The variable s denotes a history of the variation. If the domain variation is constrained on a subdomain or subboundary Θ of \bar{D} which includes these nonsmooth points, the mapping $T_s(\mathbf{X})$ is given by the identical mapping $\mathcal{S}(\mathbf{X})$ as

$$T_s(\mathbf{X}) = \mathcal{S}(\mathbf{X}) \quad \forall \mathbf{X} \in \Theta \subset \bar{D}. \quad (9)$$

The coordinate system $\mathbf{X} \in \bar{D}$ is called the Lagrange coordinate system, or the material coordinate system, and $\mathbf{x} \in \bar{D}_s$ the Euler coordinate system, or the real coordinate system.

Infinitesimal variation of the domain is given by $T_{s+\Delta s}(\mathbf{X}) = T_s(\mathbf{X}) + \Delta s \mathbf{V} + O(|\Delta s|)$, (10)

where $O(|\Delta s|)$ is defined by $\Delta s O(|\Delta s|) \rightarrow 0 (\Delta s \rightarrow 0)$. In Eq.(10), the velocity field \mathbf{V} is defined by the Euler derivative of $T_s(\mathbf{X})$ as

$$\frac{\partial T_s}{\partial s} \circ T_s^{-1}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \quad (11)$$

$$\mathbf{V} \in C_\Theta = \{ \mathbf{V} \in C^1(\bar{D}) | n_i V_i = 0 \text{ on } \partial D, \mathbf{V} = \mathbf{o} \text{ in } \Theta \subset \bar{D} \}. \quad (12)$$

The notation $z \circ y(x)$ indicates the mapping relation $x \mapsto y(x) \mapsto z(y(x))$ and $C^k(\bar{D})$ denotes the set of k class continuous functions.

The derivative of a distributed parameter can be expressed in two different ways using the Lagrange expression $\phi^s(\mathbf{X})$, $\mathbf{X} \in \Omega$, and the Euler expression $\phi_s(\mathbf{x})$, $\mathbf{x} \in \Omega_s$. These are related to $T_s(\mathbf{X})$ as

$$\phi^s(\mathbf{X}) = \phi_s \circ T_s(\mathbf{X}) \quad \forall \mathbf{X} \in \Omega. \quad (13)$$

Substituting Eq. (10), we can define the Euler derivative, or the material derivative, $\dot{\phi}_s$ and a shape derivative ϕ'_s . These are related by

$$\dot{\phi}_s = \phi'_s + \phi_{s,i} V_i, \quad (14)$$

where $\dot{\phi}_s$ and ϕ'_s are defined as

$$\dot{\phi}_s = \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} (\phi^{s+\Delta s} - \phi^s) \quad (15)$$

$$\phi'_s = \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} (\phi_{s+\Delta s} - \phi_s). \quad (16)$$

The derivatives of functionals are obtained as follows. In the case of a functional J of a distributed function ϕ_s over a domain Ω_s :

$$J = \int_{\Omega_s} \phi_s dx. \quad (17)$$

The derivative \dot{J} is given using Eq. (10) as

$$\begin{aligned} \dot{J} &= \int_{\Omega_s} \dot{\phi}_s dx + \int_{\Gamma_s} \phi_s v_n d\Gamma \\ &= \int_{\Omega_s} (\phi'_s + \phi_s V_{i,i}) dx, \end{aligned} \quad (18)$$

where $v_n = n_i V_i$.

In the case of a functional J of a distributed function ϕ_s over a boundary Γ_s :

$$J = \int_{\Gamma_s} \phi_s d\Gamma. \quad (19)$$

The derivative \dot{J} is given by

$$\begin{aligned} \dot{J} &= \int_{\Gamma_s} \{ \dot{\phi}_s + (\phi_{s,i} n_i + \phi_s \kappa) v_n \} d\Gamma \\ &= \int_{\Gamma_s} (\dot{\phi}_s + \phi_s \kappa v_n) d\Gamma, \end{aligned} \quad (20)$$

where κ denotes the curvature when Ω_s is a two-dimensional domain and the mean curvature when Ω_s is a three-dimensional domain.

4. Mean Compliance Minimization

We consider linear elastic continua with the mean compliance as an objective functional. As shown below, these are self-adjoint problems such that the sensitivity functions, or the shape gradient functions, are given with only the state variable function that is the displacement.

4.1 Formulation

The mean compliance minimization problem is formulated as follows. We assume that the distributed functions \mathbf{C} , \mathbf{f} , \mathbf{h} and \mathbf{P} determined uniquely. As examples, we consider the following conditions.

$$\text{Fixing in space : } (\dot{\cdot}) = (\cdot)_{,i} V_i ; (\dot{\cdot})' = \mathbf{o} \quad \text{in } \bar{\Omega} \quad (21)$$

$$\text{Fixing in material : } \begin{cases} (\dot{\cdot}) = \mathbf{o} ; (\dot{\cdot})' + (\cdot)_{,i} V_i = \mathbf{o} & \text{in } \Omega \\ (\dot{\cdot}) = \mathbf{o} ; (\dot{\cdot})' + (\cdot)_{,i} n_i v_n = \mathbf{o} & \text{on } \Gamma \end{cases} \quad (22)$$

$$\text{Covariation with material : } \begin{cases} (\dot{\cdot}) + (\cdot) V_{i,i} = \mathbf{o} ; \\ (\dot{\cdot})' + \{ (\cdot) V_i \}_{,i} = \mathbf{o} & \text{in } \Omega \\ (\dot{\cdot}) + (\cdot) \kappa v_n = \mathbf{o} ; \\ (\dot{\cdot})' + \{ (\cdot)_{,i} n_i + (\cdot) \kappa \} v_n = \mathbf{o} & \text{on } \Gamma \end{cases} \quad (23)$$

When we assume a constraint of the magnitude of the domain

$$m = \int_{\Omega_s} dx, \quad (24)$$

the mean compliance minimization problem is described as follows.

▷ Problem : Given distributed functions \mathbf{C} , \mathbf{f} , \mathbf{h} and $\mathbf{P} \in C^1(\bar{D})$ that are determined uniquely with respect to the domain variation defined in Eqs. (11) and (12) and a magnitude limit for the domain $M \in R_+$, find $\Omega_s = T_s(\Omega)$ that minimizes the mean compliance, $l(\mathbf{u})$, (25)

subject to the equilibrium equation

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}) &= l(\mathbf{w}) - l_h(\mathbf{w}) \\ \mathbf{v} &= \mathbf{u} - \mathbf{h} \in H_{r_1}, \quad \forall \mathbf{w} \in H_{r_1}, \end{aligned} \quad (26)$$

and the domain magnitude constraint

$$m - M \leq 0 \quad \leq, \quad (27)$$

where R_+ is a set of positive real numbers.

4.2 Optimality criteria

Applying the Lagrange multiplier method, we derive the optimality criteria.

Using \mathbf{w} as the Lagrange multiplier for Eq. (26) and Λ as the Lagrange multiplier for the inequality (27), the above problem can be rendered into the stationalization problem of the Lagrange functional

$$L = l(\mathbf{v}) - l_h(\mathbf{v}) - a(\mathbf{v}, \mathbf{w}) + l(\mathbf{w}) - l_h(\mathbf{w}) + \Lambda(m - M) \quad \mathbf{w} \in H_{r_1}, \quad \Lambda \in R_+. \quad (28)$$

The derivative \dot{L} of the Lagrange functional L with respect to the domain variation is derived using the formula in section 3 as follows.

$$\begin{aligned} \dot{L} = & \int_{\Omega_s} (f'_i v_i + f'_i v'_i) dx + \int_{\Gamma_s} f_i v_i n_j V_j d\Gamma \\ & + \int_{\Gamma_{s2}} \{P'_i v_i + P_i v'_i \\ & + (P_{i,j} n_j v_i + P_i v_{i,j} n_j + P_i v_i \boldsymbol{\chi}) n_k V_k\} d\Gamma \\ & - \int_{\Omega_s} (C'_{ijkl} h_{k,l} v_{i,j} + C_{ijkl} h'_{k,l} v_{i,j} \\ & + C_{ijkl} h_{k,l} v'_{i,j}) dx - \int_{\Gamma_s} C_{ijkl} h_{k,l} v_{i,j} n_m V_m d\Gamma \\ & - \int_{\Omega_s} (C'_{ijkl} v_{k,l} w_{i,j} + C_{ijkl} v'_{k,l} w_{i,j} \\ & + C_{ijkl} v_{k,l} w'_{i,j}) dx - \int_{\Gamma_s} C_{ijkl} v_{k,l} w_{i,j} n_m V_m d\Gamma \\ & + \int_{\Omega_s} (f'_i w_i + f_i w'_i) dx + \int_{\Gamma_s} f_i w_i n_j V_j d\Gamma \\ & + \int_{\Gamma_{s2}} \{P'_i w_i + P_i w'_i \\ & + (P_{i,j} n_j w_i + P_i w_{i,j} n_j + P_i w_i \boldsymbol{\chi}) n_k V_k\} d\Gamma \\ & - \int_{\Omega_s} (C'_{ijkl} h_{k,l} w_{i,j} + C_{ijkl} h'_{k,l} w_{i,j} \\ & + C_{ijkl} h_{k,l} w'_{i,j}) dx - \int_{\Gamma_s} C_{ijkl} h_{k,l} w_{i,j} n_m V_m d\Gamma \\ & + \int_{\Gamma_s} \Lambda n_i V_i d\Gamma + \dot{\Lambda}(m - M) \\ = & l(\mathbf{v}') - l_h(\mathbf{v}') - a(\mathbf{v}', \mathbf{w}) + l(\mathbf{w}') \\ & - l_h(\mathbf{w}') - a(\mathbf{v}, \mathbf{w}') + \dot{\Lambda}(m - M) + l_G(\mathbf{V}), \quad (29) \end{aligned}$$

where

$$\begin{aligned} l_G(\mathbf{V}) = & \int_{\Gamma_s} \{f'_i (v_i + w_i) - C'_{ijkl} (v_{k,l} w_{i,j} + h_{k,l} v_{i,j} \\ & + h_{k,l} w_{i,j}) + \Lambda\} n_m V_m d\Gamma \\ & + \int_{\Omega_s} \{f'_i (v_i + w_i) - C'_{ijkl} (v_{k,l} w_{i,j} + h_{k,l} v_{i,j} \\ & + h_{k,l} w_{i,j}) + C_{ijkl} (h'_{k,l} v_{i,j} + h'_{k,l} w_{i,j})\} dx \\ & + \int_{\Gamma_{s2}} \{P'_i v_i + P_i v'_i + (P_{i,j} n_j v_i + P_i v_{i,j} n_j \\ & + P_i v_i \boldsymbol{\chi} + P_{i,j} n_j w_i + P_i w_{i,j} n_j \\ & + P_i w_i \boldsymbol{\chi}) n_k V_k\} d\Gamma. \quad (30) \end{aligned}$$

Under the assumption that \mathbf{C} , \mathbf{f} , \mathbf{h} and \mathbf{P} are determined uniquely, we recognize that $l_G(\mathbf{V})$ is a linear form of the velocity field \mathbf{V} given by

$$l_G(\mathbf{V}) = \int_{\Omega_s} G_i V_i dx. \quad (31)$$

The coefficient vector function \mathbf{G} is the sensitivity of the velocity field to the objective functional and is called the shape gradient function. Now we derive expressions for \mathbf{G} for the case in which \mathbf{C} , \mathbf{f} and \mathbf{h}

are fixed in space and $\mathbf{h} = \mathbf{o}$ in \bar{D} . When \mathbf{P} is fixed in space, that is $\mathbf{P}' = \mathbf{o}$, \mathbf{G} is given by

$$\begin{aligned} \mathbf{G} = & \gamma_{\Gamma_s}((f_i u_i + f_i w_i - C_{ijkl} u_{k,l} w_{i,j} + \Lambda) \mathbf{n}) \\ & + \gamma_{\Gamma_{s2}}((P_{i,j} n_j u_i + P_i u_{i,j} n_j + P_i u_i \boldsymbol{\chi} \\ & + P_{i,j} n_j w_i + P_i w_{i,j} n_j + P_i w_i \boldsymbol{\chi}) \mathbf{n}), \quad (32) \end{aligned}$$

where $\gamma_{\Gamma_s}(\phi)$ is the trace operator projecting ϕ defined in Ω_s to its boundary Γ_s . When \mathbf{P} is fixed in the material, $\dot{\mathbf{P}} = \mathbf{o}$, and

$$\begin{aligned} \mathbf{G} = & \gamma_{\Gamma_s}((f_i u_i + f_i w_i - C_{ijkl} u_{k,l} w_{i,j} + \Lambda) \mathbf{n}) \\ & + \gamma_{\Gamma_{s2}}((P_i u_{i,j} n_j + P_i u_i \boldsymbol{\chi} + P_i w_{i,j} n_j \\ & + P_i w_i \boldsymbol{\chi}) \mathbf{n}). \quad (33) \end{aligned}$$

If \mathbf{P} covaries with the material, then $\dot{\mathbf{P}} + \mathbf{P} \boldsymbol{\chi} v_n = \mathbf{o}$, and

$$\begin{aligned} \mathbf{G} = & \gamma_{\Gamma_s}((f_i u_i + f_i w_i - C_{ijkl} u_{k,l} w_{i,j} + \Lambda) \mathbf{n}) \\ & + \gamma_{\Gamma_{s2}}((P_i u_{i,j} n_j + P_i w_{i,j} n_j) \mathbf{n}). \quad (34) \end{aligned}$$

In the three cases above, \mathbf{G} is given as a distributed function of the normal vector on the boundary Γ_s .

From Eq. (29), the stationality conditions of the Lagrange function L are

$$a(\mathbf{v}, \mathbf{w}') = l(\mathbf{w}') - l_h(\mathbf{w}') \quad \forall \mathbf{w}' \in H_{r_1} \quad (35)$$

$$a(\mathbf{v}', \mathbf{w}) = l(\mathbf{v}') - l_h(\mathbf{v}') \quad \forall \mathbf{v}' \in H_{r_1} \quad (36)$$

$$l_G(\mathbf{V}) = 0 \quad \forall \mathbf{V} \in C_\theta \quad (37)$$

$$\Lambda(m - M) = 0 \quad (38)$$

$$m - M \leq 0. \quad (39)$$

Under these conditions, Eq.(35) is the equation governing \mathbf{v} that agrees with Eq. (1). Using Eq. (36), we can determine \mathbf{w} from the relation

$$\mathbf{v} = \mathbf{w}. \quad (40)$$

In general, \mathbf{w} is called an adjoint function and Eq. (36) an adjoint equation. Since the state equation and the adjoint equation agree, then this problem is known as a self-adjoint problem. Equations (38) and (39) are parts of the Kuhn-Tucker conditions relating to the inequality condition of Eq. (27). The Lagrange multiplier Λ can be determined using these conditions.

Considering the relations $\mathbf{v} = \mathbf{w}$ determined from Eq. (35) and Λ determined from Eqs. (38) and (39), the derivative of the Lagrange function \dot{L} , which agrees with the derivative of the objective function with respect to the velocity field \mathbf{V} , can be expressed as

$$\dot{L} = l_G(\mathbf{V}). \quad (41)$$

5. Traction Method

Since the derivative of the Lagrange functional \dot{L} is obtained as a linear form of the velocity field \mathbf{V} with a coefficient function of \mathbf{G} , we can apply the traction method to the compliance minimization problem. First we describe the traction method concisely⁽⁴⁾.

We consider the k -th domain variation with the velocity field $\mathbf{V}^{(k)}$. The traction method is used to determine $\mathbf{V}^{(k)}$ by

$$a(\mathbf{V}^{(k)}, \mathbf{w}) = -l_c^{(k)}(\mathbf{w}) \quad \mathbf{V}^{(k)} \in C_\theta \quad \forall \mathbf{w} \in C_\theta. \quad (42)$$

We can confirm that a domain variation with velocity field $\mathbf{V}^{(k)}$ decreases the Lagrange functional L as follows. When the state equation (35) and the Kuhn-Tucker conditions (38) and (39) are satisfied, the perturbational expansion of the Lagrange functional L around the k -th mapping $T_{s^{(k)}}(\mathbf{X})$ is given by $\Delta L^{(k)} = l_c^{(k)}(\Delta s \mathbf{V}^{(k)}) + O(|\Delta s|)$. (43)

Substituting Eq. (42) into Eq. (43) and considering the positive definiteness of $a(\mathbf{v}, \mathbf{w})$ based on Eq. (7), then $\exists \alpha > 0 : a(\xi, \xi) \geq \alpha \|\xi\|^2 \quad \forall \xi \in (H^1(\Omega_s))^n$. (44)

The following relation is obtained for a sufficiently small value of Δs .

$$\Delta L^{(k)} = -a(\Delta s \mathbf{V}^{(k)}, \Delta s \mathbf{V}^{(k)}) < 0 \quad (45)$$

This equation shows that varying the domain with velocity field $\mathbf{V}^{(k)}$ obtained from Eq. (42) decreases the Lagrange functional L , or the objective function, in case in which the convexity and boundness are ensured.

Equation (42) indicates that the velocity field $\mathbf{V}^{(k)}$ is obtained as a displacement of the pseudoelastic body defined in Ω_s due to the loading of a pseudoexternal force in proportion to $-\mathbf{G}$ under constraints on the displacement of the invariable subdomains or subboundaries as shown in Fig. 3. To solve the pseudoelastic problem, we can use any numerical analysis method applicable to linear elastic problems, such as the finite element or the boundary element method. In this work, the FEM was used.

The Lagrange multiplier Λ that satisfies the Kuhn-Tucker conditions (38) and (39) is determined as follows. Since $\Lambda \mathbf{n}$ contributes to the pseudoforce $-\mathbf{G}$ as a uniform boundary force, the relationship among the variation of the uniform boundary force $\Delta \Lambda \mathbf{n}$, the variation of the velocity field $\Delta \mathbf{V}$ and the variation of the magnitude of the domain Δm is obtained by elastic deformation analysis based on the following equation with a uniform boundary force $\Delta \Lambda \mathbf{n}$ as shown in Fig. 4.

$$a(\Delta \mathbf{V}, \mathbf{w}) = \Delta \Lambda \int_{\Gamma_s} \mathbf{n}_i w_i d\Gamma \quad \Delta \mathbf{V} \in C_\theta \quad \forall \mathbf{w} \in C_\theta \quad (46)$$

$$\Delta m = \int_{\Gamma_s} \mathbf{n} \cdot \mathbf{V} d\Gamma \quad (47)$$

Based on the linearity between $\Delta \mathbf{V}$ and Δm for small deformations, we derive the following renewal procedure for Λ and \mathbf{V} .

$$\Lambda_{(new)} = \begin{cases} \max \left[0, \Lambda_{(old)} + \Delta \Lambda \frac{m_{(old)} - M}{\Delta m} \right] & (\Lambda_{(old)} > 0, m_{(old)} - M \neq 0) \\ \Delta \Lambda \frac{m_{(old)} - M}{\Delta m} & (\Lambda_{(old)} = 0, m_{(old)} - M > 0) \\ 0 & (\Lambda_{(old)} = 0, m_{(old)} - M \leq 0) \end{cases} \quad (48)$$

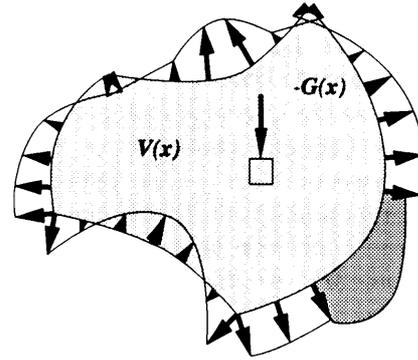


Fig. 3 Traction method

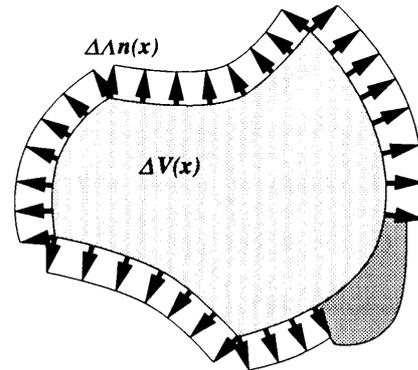


Fig. 4 Uniform traction

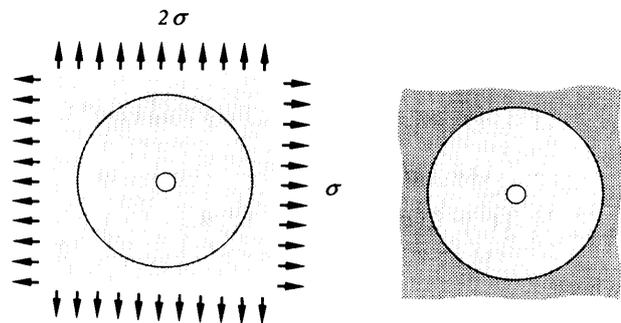


Fig. 5 Infinite plate with a hole: elastic deformation (left) / domain variation (right)

$$\mathbf{V}_{(new)} = \mathbf{V}_{(old)} \frac{\Lambda_{(new)} - \Lambda_{(old)}}{\Delta \Lambda} \quad (49)$$

6. Numerical Analyses

We present results of some numerical analyses for basic problems using the traction method and shape gradient functions derived in section 4.

6.1 Infinite plate with a hole

As a domain optimization problem for which the analytical solution is known, we applied the traction method to the plane stress in an infinite plate with a hole loaded with a traction of 1 : 2 perpendicularly at infinite distance as shown in Fig. 5. The optimal shape of the hole is known to be an ellipse with axes of 1 :

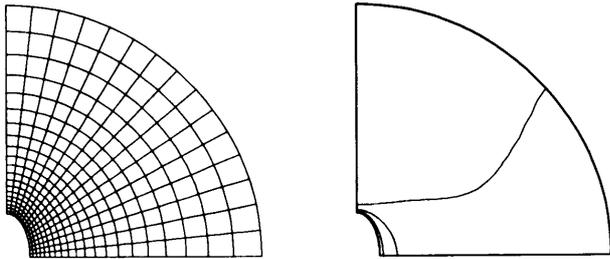


Fig. 6 Infinite plate with optimized hole: finite element mesh (left)/ strain energy density (right)

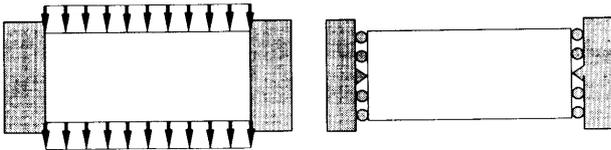


Fig. 7 Plate clamped at both ends: elastic deformation (left)/ domain variation (right)

r in the case of perpendicular loading of 1 : $r^{(7)}$. For the numerical analysis, we chose the first quadrant with a suitable radius considering the symmetry of the problem. The traction on the outer arc was determined from the analytical solution for a plate with a circular hole. The velocity field was analyzed under the constraint on the outer arc. Since the hole is traction-free, the shape gradient function \mathbf{G} was calculated using only the $\gamma_{r,(\cdot)}$ term in Eq. (32), where $\mathbf{f} = \mathbf{o}$. In the numerical analyses, we used eight-node isoparametric elements.

The result obtained by starting with a circular hole is shown in Fig. 6. This result agrees with the analytical solution for an elliptical hole with axes of 1 : 2. We confirmed that the results obtained for other initial shapes converged to the same shape. The iteration number, which depends on the value of Δs , was 5 in the case shown in Fig. 6.

6.2 Plate clamped at both ends

To confirm the validity of the traction method using shape gradient functions when the loaded boundary varies, we applied it to a simple plate clamped at both ends as shown in Fig. 7. The traction force \mathbf{P} applied on the upper and lower boundaries was assumed to have a downward and uniform distribution at the initial boundary. Then, the traction fixed in space at $\mathbf{P}' = \mathbf{o}$ agrees with that fixed in the material at $\dot{\mathbf{P}} = \mathbf{o}$. In this case, the traction is applied uniformly on the boundaries while the domain varies. In contrast, when we assume that the traction covaries with the material, i.e., $\dot{\mathbf{P}} + \mathbf{P}\kappa v_n = \mathbf{o}$, the magnitude of the traction varies in inverse proportion to the expansion ratio of the applied boundary.

The optimized results are shown in Fig. 8 for the

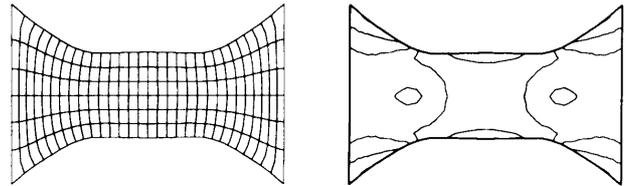


Fig. 8 Optimized plate clamped at both ends loaded with traction fixed in space $\mathbf{P}' = \mathbf{o}$ or fixed in the material $\dot{\mathbf{P}} = \mathbf{o}$: finite element mesh (left)/ strain energy density (right)

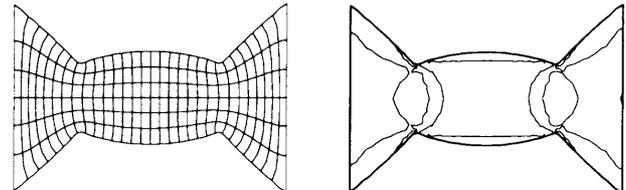


Fig. 9 Optimized plate clamped at both ends loaded with traction which covaries with the material $\dot{\mathbf{P}} + \mathbf{P}\kappa v_n = \mathbf{o}$: finite element mesh (left)/ strain energy density (right)

case in which the traction is fixed in space, $\mathbf{P}' = \mathbf{o}$, and that in which it is fixed in the material $\dot{\mathbf{P}} = \mathbf{o}$. Figure 9 shows the case in which the traction covaries with the material, $\dot{\mathbf{P}} + \mathbf{P}\kappa v_n = \mathbf{o}$. The shape gradient functions \mathbf{G} are calculated using Eqs. (33) and (34), respectively. In both cases, we confirmed that the mean compliances of the converged results obtained using the correct shape gradient functions are smaller than those obtained using other shape gradient functions. The results verify the validity of the traction method using shape gradient functions derived in section 4.

7. Conclusion

In this study, we derived shape gradient functions for mean compliance minimization problems theoretically, allowing variation of the loaded boundary, the body force and the elastic stiffness. The validity of the traction method using the derived shape gradient functions was confirmed by the agreement of the converged shape with the analytical solution in the case of an infinite plate with a hole, and by comparison between the mean compliances of converged results obtained using different shape gradient functions in the case of a plate clamped at both ends with varying load boundaries.

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