Optimization of Truss Topology Using Boundary Cycle*
(Derivation of Design Variables to Avoid Inexpedient Structure)

Yasuhiko NAKANISHI** and Shigeru NAKAGIRI***

This paper deals with optimization of truss topology using boundary cycle in algebraic topology. Elimination of unnecessary members from the ground structure, one of the popular means to optimize truss topology, is employed. The elimination has a disadvantage that unstable structures possibly appear in the process of the optimization. Boundary operator, which has the ability to represent equilibrium of internal force in members, is used to generate the boundary cycle from chain. Design variables derived by the boundary cycle can always satisfy this equilibrium and avoid a category of unstable structures without imposing any constraint. An attempt is made through numerical examples to minimize the total weight of a plane truss, which is fixed to a rigid wall and supports a vertical load acting at a point distant from the wall, under the condition that the distribution of strain energy density is uniform and equal to a certain value. The validity of this formulation is verified by the numerical examples concerned with the weight minimization of the truss.

Key Words: Optimum Design, Computational Mechanics, Finite Element Method, Topology, Boundary Cycle, Truss

1. Introduction

Topological condition of the optimum structure is evident in some structural optimization problems on the basis of engineering judgment. For example, members of a frame with a tip disconnected from other members are useless, because the members do not carry any load and their weight is nothing but burden to the frame. Topology of the optimum structure should be represented as a set of loops, when such useless members have to be avoided. The purpose of this paper is to derive design variables that satisfy such the topological condition. Some numerical examples have revealed that mass of the planar trusses fixed on a rigid wall and loaded vertically on a point distant from the wall can be minimized by means of eliminating unnecessary members from a ground structure[1]. The necessary condition for an optimum truss includes those for an optimum frame and the satisfaction of equilibrium of internal forces, concentrated loads and reaction forces at the support. The equilibrium condition makes optimization of trusses more difficult than that of frames. In this paper, structure of a truss is represented as a set of loops, and design variables to avoid such structures that cannot satisfy the equilibrium of force are derived on the basis of the boundary cycle[2),(3] used in algebraic topology. The cross section of the truss members is supposed to be circular, and the design variables are assigned to the diameter of the cross section, because the mutual relationship between the cross sections can be controlled so that the force equilibrium is satisfied during the optimization.

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** Department of Mechanical Engineering, Toyohashi University of Technology, 1-1 Hibarigaoka, Tempaku-cho, Toyohashi 441, Japan

*** Institute of Industrial Science, University of Tokyo, 7-22-1 Roppongi, Minato-ku, Tokyo 106, Japan

2. Statement of Problem

Figure 1 shows ground structures (a)～(e) used in optimization of a planar truss. The left side of each ground structure is fixed to a rigid wall, and a point A on the right side sustains a concentrated vertical load of 9.8 kN. All the members are assumed to be made

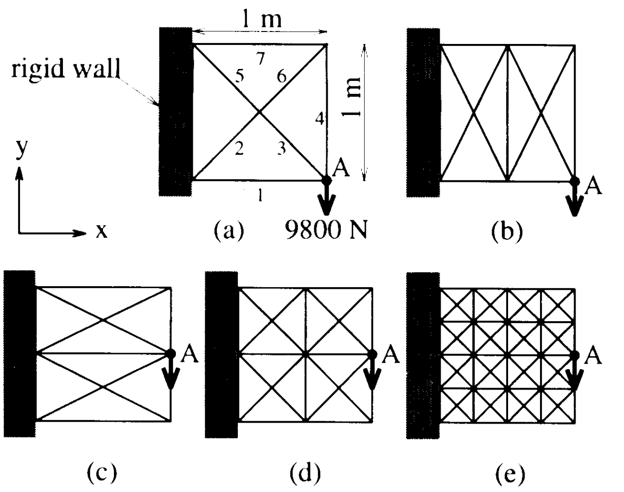

Fig. 1 Ground structures

of steel and have circular cross section. The design domain is set as an inner space of a square of 1 m in span. Such a constraint condition is employed that the strain energy density in the truss is distributed uniformly equal to a prescribed value in all the members. In other words, the axial stress in each member is set equal to a certain constant. Then the design under this condition is a kind of fully stressed design. The strain energy density is calculated due to infinitesimal displacement theory in elastic problems. Design variables are chosen so as to avoid only structures which cannot satisfy the force equilibrium. Consequently, a structure is considered candidate for the optimization, even when it is statically indeterminate or unstable in case if it satisfies the force equilibrium.

3. Definition of Boundary Cycle

Boundary cycle can be utilized to derive parameters which always satisfy the force equilibrium, and those parameters are called design parameters in this study. The definition of boundary cycle is summarized as follows, while the detail is seen in the Ref. (4).

r-simplex x^r is written as $(a_0, a_1, \cdots, a_r)$ where each $a_i (i=1, \cdots, r)$ is an independent vertex. A set of simplexes is called simplicial complex. r chain c^r is defined as sum of $t^i x_i^r (i=1, \cdots, m)$,

$$c^r = t^1 x_1^r + t^2 x_2^r + \cdots + t^m x_m^r \quad (1)$$

where m is the number of r simplexes belonging to the simplicial complex. Each coefficient $t^i (i=1, \cdots, m)$ must be an element of the Abelian group[5] to define the boundary cycle. Though t^i is, in general, treated as an integer coefficient, t^i in this paper is used as a real number to represent a dimension of cross section of each member. The boundary $(r-1)$ cycle $\partial_r c^r$ is generated by operating ∂_r on r chain c^r. ∂_r is called boundary operator and defined as

$$\partial_r x^r = \sum_{i=0}^{r} (-1)^i (a_0, \cdots, \hat{a}_i, \cdots, a_r) \quad (2)$$

where $\hat{a}_i$ expresses lack of a vertex a_i. The boundary operator ∂_r has an important feature that

$$\partial_{r-1}(\partial_r x^r) = 0 \quad (3)$$

for any x^r.

4. Optimization of Topology and Shape by Boundary Cycle

4.1 Derivation of design parameters

Two simplicial complexes related to a ground structure are chosen for representation of the force equilibrium. These two simplicial complexes express the x-direction and y-direction components of forces in a truss, respectively. A simplicial complex related to a certain ground structure is not unique. Figure 2 shows an example of a simplicial complex related to the ground structure (a) shown in Fig. 1. This simplicial complex consists of 2-simplexes (triangles), 1 simplexes (line segments) and 0 simplexes (points). Coefficients of these simplexes express the equilibrium of the x-direction components of the forces. This simplicial complex is explained in the following by use of the numbers given in Fig. 2.

Coefficients of 1 simplexes $f^i (i=1, \cdots, 12)$ depend on those of 2 simplexes $t^i (i=1, \cdots, 7)$. For example, a coefficient of 1 simplex 2 is decided by coefficients of 2 simplexes ① and ②. 2 simplexes ⑤, ⑥ and ⑦ are virtual 2 simplexes placed to represent the x direction components of the reaction force in the rigid wall and the vertical load on the point A. Coefficients of 1 simplexes 1~7, 9~10 and 12 correspond to the x direction components of the internal force in the members, the reaction force and the external load, respectively. 1 simplexes 8 and 11 are set to make virtual 2 simplexes for the sake of convenience. Equation (3) guarantees that all the coefficients of 0 simplexes are equal to zero. A coefficient of 0 simplex is sum of the coefficients of 1 simplexes (internal force in the members) connected to the 0 simplex. As the result, the force equilibrium is satisfied by means of making the coefficients of 0 simplexes equal to zero. The reaction force and the load are balanced at point B. Another simplicial complex to represent the y-direction components of the forces can be formulated in the same way.

As for the simplicial complex of the x direction components of the forces, let $\{f\}$ and $\{t\}$ be the vector whose elements are coefficients of 1 simplexes and 2 simplexes, respectively, and let $[A]$ be a matrix determined by the boundary operator ∂_2 and the ground structure. $\{f\}$ is related to $\{t\}$ linearly through $[A]$ as follows.

$$\{f\} = [A]\{t\} \quad (4)$$

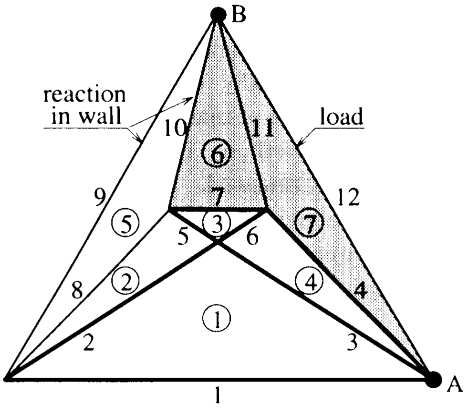

Fig. 2 Example of simplicial complex related to ground structure (a)

The ingredients of $[A]$ take one of -1, 0 or 1. A similar equation to Eq. (4) holds for the simplicial complex of the y direction components of the forces, that is, Eq. (5), where let $\{g\}$ and $\{u\}$ denote the vector equivalent to $\{f\}$ and $\{t\}$, respectively.

$$\{g\}=[A]\{u\} \tag{5}$$

$\{f\}$ and $\{g\}$ are not arbitrary vector, and there is a relationship between them governed by the direction of the member axes, because the orientation of an internal force in a member must be identical to the member axis. The orientation of the reaction force in the wall is not restricted, so that Eqs. (6) and (7) are significant which are constituted by extracting only the rows concerning with the internal forces and vertical load from Eqs. (4) and (5) and indicated by suffix d.

$$\{f_d\}=[A_d]\{t\} \tag{6}$$

$$\{g_d\}=[A_d]\{u\} \tag{7}$$

The relationship between $\{f_d\}$ and $\{g_d\}$ is described by a diagonal matrix $[T]$, whose ingredients are tangent of the member axes.

$$\{g_d\}=[T]\{f_d\} \tag{8}$$

The axes of coordinate should be properly rotated in order to prevent the ingredients of $[T]$ from being infinite. Equation (9) is obtained by substituting Eqs. (6) and (7) into the vectors in Eq. (8).

$$[A_d]\{u\}=[T][A_d]\{t\} \tag{9}$$

The existence condition of solutions of $\{u\}$ is

$$([I]-[A_d][A_d]^-)[T][A_d]\{t\}=\{0\} \tag{10}$$

where $[I]$ is an identity matrix, and $[A]^-$ is the Moor-Penrose generalized inverse[6] of $[A]$. Equation (10) being rewritten as

$$[B]\{t\}=\{0\}, \tag{11}$$

the general solution of $\{t\}$ is given by the complementary solution as Eq. (12) because the particular solution of Eq. (11) is $\{0\}$,

$$\{t\}=([I]-[B]^-[B])\{h\} \tag{12}$$

where $\{h\}$ is an arbitrary vector. The vector $\{t\}$ in Eq. (12) always satisfys the force equilibrium. When $\{f\}$ is determined by Eqs. (4) and (12), and $f^i=0$ means the elimination of the i-th member corresponding to f^i, topology of a truss except members with a loading or supporting point is always expressed as a set of loops.

Since magnitude of the load is not included in Eq. (12), $\{h\}$ should be constrainted so that the orientations of 2-simplexes ⑥ and ⑦ are the same, and that their coefficients are equal to the magnitude of the x-direction component of the load. The orientations of these 2-simplexes depend upon the load direction, because combination of the orientation of 1-simplexes No. 12 and these 2-simplexes decides sign of the x-direction component of the load. Let $\{t_c\}$ be a vector consisting of the coefficients of 2-simplexes ⑥ and ⑦, and $[C]$ be a matrix made of rows related with $\{t_c\}$ extracted from $[I]-[B]^-[B]$ in Eq. (12). Then, the following equation must hold between $\{t_c\}$ and $\{h\}$.

$$\{t_c\}=[C]\{h\} \tag{13}$$

When $\{t_c\}=P_x\{1\}$, the general solution of $\{h\}$ becomes

$$\{h\}=P_x[C]^-\{1\}+([I]-[C]^-[C])\{p\} \tag{14}$$

where P_x means the magnitude of the x-direction component of the load, $\{1\}$ is a vector whose ingredients are 1, and $\{p\}$ is an arbitrary vector. The constraint on $\{h\}$ guarantees that more than one loop surrounding 2-simplexes ⑥ and ⑦ exist. Consequently, all trusses thus generated by the above formulation always connect the loading point A with the rigid wall.

Finally, the vector of the coefficients of 1-simplexes $\{f\}$ is written as follows using Eqs. (4), (12) and (14).

$$\{f\}=\{f_a\}+[F_b]\{p\} \tag{15}$$

where

$$\{f_a\}=P_x[A]([I]-[B]^-[B])[C]^-\{1\},$$

$$[F_b]=[A]([I]-[B]^-[B])([I]-[C]^-[C]).$$

When the ingredients of $\{p\}$ are treated as design parameter, $\{f\}$ due to Eq. (15) can always satisfy the force equilibrium without imposing any constraint on $\{p\}$. However, this expression does not mean the uniqueness of displacements in trusses.

The formulation described in this section is summarized as given below. A coefficient of a certain 0-simplex is the sum of the coefficients of 1-simplexes connected with the 0-simplex and corresponds to the resultant internal force at the node (0-simplex). Moreover, Eq. (3) guarantees that the coefficients of all 0-simplexes are zero. The resultant internal force represented by $\{f\}$ and $\{g\}$ is always held equal to zero at each node, i.e. the force equilibrium is always satisfied. As the result, Eq. (3) guarantees that $\{f\}$ in Eq. (15) satisfies the force equilibrium.

4.2 Objective function

An objective function F dealt with herein is mass of a truss. F is written as a function of the number of truss members n and diameter of cross section of the i-th member d_i as given below.

$$F=\sum_{i=1}^{n}\frac{\pi}{4}\rho l_i d_i^2 \tag{16}$$

where l_i is length of the i-th member, and ρ is mass density of the material, both of them being taken constant. The diameter of cross section d_i is formulated as Eq. (17) by the constraint condition that strain energy density must be uniform and equal to prescribed value U,

$$d_i=\sqrt[4]{\frac{8\{(f^i)^2+(g^i)^2\}}{\pi^2 EU}} \tag{17}$$

where $\sqrt{(f^i)^2+(g^i)^2}$ is the internal force in the i-th member and E is the Young's modulus. This objective function is also a function of the design parameters $\{p\}$ in Eq. (15). Substitution of Eq. (17) into d_i in Eq. (16) generates the following result.

$$F=\sum_{i=1}^{n}\frac{\rho l_i}{\sqrt{2EU}}\sqrt{(f^i)^2+(g^i)^2} \tag{18}$$

In this formula, the internal force indirectly plays a role of design variables. In general, the internal force in the optimization has different value from actual truss except the optimum truss. In other words, since the constraint condition concerning with strain energy density is satisfied for only the internal force supposed in the optimization, the strain energy density in a truss obtained by analysis will be inconsistent with the prescribed value U at a stage of the optimization process. Mass of a truss is closely related with total strain energy, and then the minimization of mass is essential for identification of the strain energy density obtained by the analysis with U.

4.3 Method to minimize mass

The objective function F in Eq. (18) forms a convexity consisting of hyperplanes in the space of design parameters and F. Minimization of the function F can be recognized as a linear programming problem. The number of hyperplanes in the convexity is equivalent to the number of combinations of plus and minus signs of the x direction components of the internal force. For example, the number of hyperplanes of the ground structure (e) which has 100 members, i.e. the largest number among (a)~(e), is estimated as enormous number of $2^{100}=1.26\times10^{30}$. Though actual number dealt with in the optimization will be smaller than 2^{100}, the number of hyperplanes is still too large to employ the simplex method. Therefore, the Newton-Raphson's method is utilized to search the minimum of F. The following method is utilized, since the objective function F is linear to the design parameters, those methods that use only sensitivity are not proper to be applied in the search. First, a target value of mass slightly smaller than the current value is set. Secondly, the design parameters to realize the target value of mass are determined by the Newton-Raphson's method. Thirdly, the target value of smaller mass is renewed. The second and third steps mentioned above are repeated successively, and eventually the minimum mass will be found. This method is hardly affected by the number of hyperplanes.

5. Numerical Examples

The mass of truss is minimized based on the ground structures (a)~(e). The prescribed value of strain energy density U, Young's modulus E and mass density ρ are taken equal to 5.95×10^3 J/m^3, 202 GPa and 7.83×10^3 kg/m^3, respectively. These values are equivalent to set the absolute values of stress in all the members equal to 49.0 MPa.

Figure 3 shows the relation between a design parameter p (the abscissa) and mass of a structure (the ordinate) obtained from the ground structure (a). In this chapter the number of design parameters is reduced to make the aforementioned steps efficient by means of eigenvalue decomposition of a matrix $[F_b]$ in Eq. (15). In the case of (a), the number of design parameters is decreased to one. The numbers written beside the right side of Fig. 3 correspond to the number of the members shown by the upper side of the same figure. The solid line which is obtained by the sum of two broken lines representing mass of each member is the trace of the objective function F.

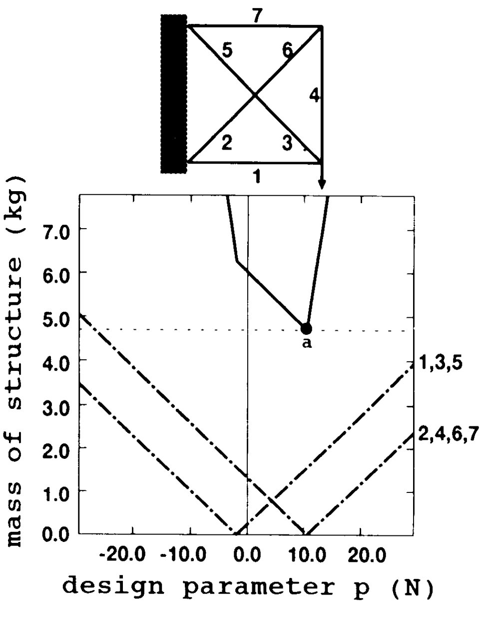

Fig. 3 Relation between design parameter and mass of structure

Three members 1, 3 and 5, whose mass can be expressed by the same function (one of the broken lines), are eliminated or left at the same time in the optimization. On the other hand, four members 2, 4, 6 and 7 correspond to another broken line. It should be noted that the members belonging to different broken lines cannot be eliminated from (a) at once. As for the members 1 and 2, at least one of them always remains. A point **a** in Fig. 3 denotes the optimum structure shown in Fig. 4. This structure obtained by elimination of the members 2, 4, 6 and 7 from (a) has two diagonal members. Displacement at a point that connects these members cannot be determined uniquely owing to the assumption that the deformation is infinitesimally small. It is desirable in design of actual truss to merge these members in one piece.

The eigenvalue decomposition of the matrix $[F_b]$ for the ground structure (b) reduces the number of design parameters to two. The abscissa and the ordinate in Fig. 5 are design parameters p_1 and p_2, respectively. Numbers on the right hand of Fig. 5 indicate the number of the members shown by the upper side. When combination of p_1 and p_2 satisfies relation represented by a solid line, mass of members concerning the solid line is zero. The members shown by parallel solid lines, such as members 1 and 4, cannot vanish at the same time. Figure 6 shows the optimum structure corresponding to the point **a** in Fig. 5. Though the ground structure (b) has a larger number of members than the ground structure (a), it gives rise to a heavier truss than (a). It is shown that a ground structure which has a larger number of members does not always generate a better structure.

Figure 7 shows the optimum structure obtained from the ground structure (c). This truss has two pairs of diagonal member inclined approximately 63.4° to the wall, and this is the lightest one among the optimum trusses found in this paper (see Table 1).

Figure 8 depicts the optimum structure based on

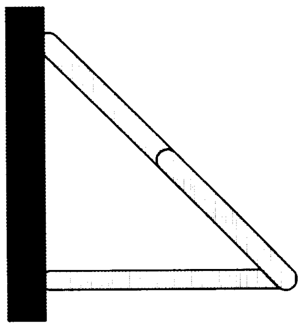

Fig. 4 Optimum structure obtained from ground structure (a)

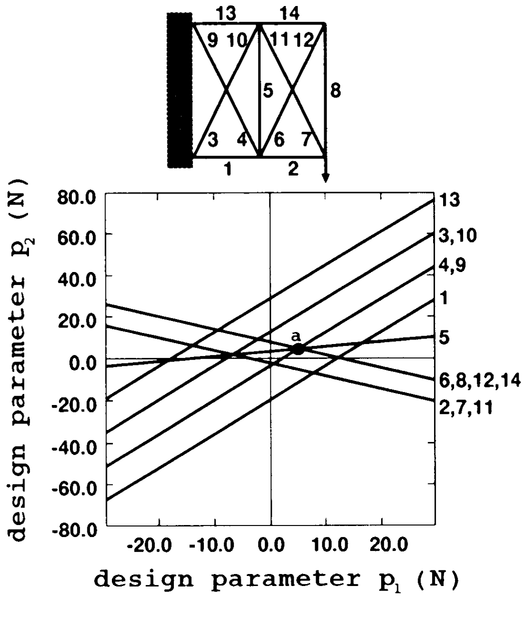

Fig. 5 Relation between design parameters p_1 and p_2

Table 1 Mass of optimum structures

figure	ground structure	mass of structure (kg)
Fig. 4	(a)	4.70
Fig. 6	(b)	5.48
Fig. 7	(c)	3.92
Fig. 8, 9	(d)	4.70
Fig. 10	(d)	6.27
Fig. 11, 12	(e)	4.70

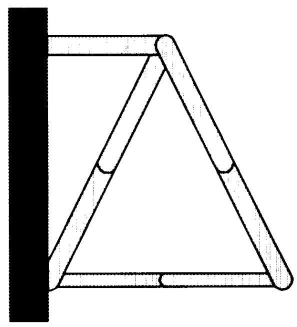

Fig. 6 Optimum structure obtained from ground structure (b)

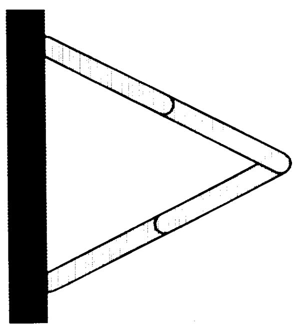

Fig. 7 Optimum structure obtained from ground structure (c)

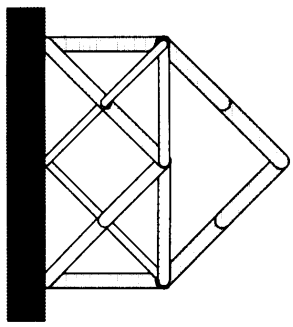

Fig. 8 Optimum structure obtained from ground structure (d)(I)

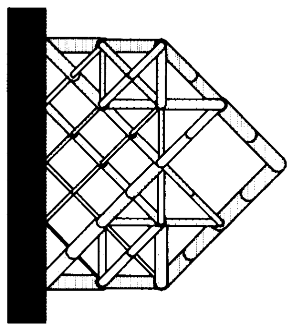

Fig. 11 Optimum structure obtained from ground structure (e) (I)

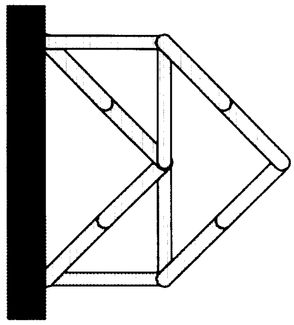

Fig. 9 Optimum structure obtained from ground structure (d)(II)

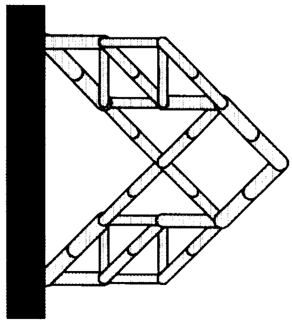

Fig. 12 Optimum structure obtained from ground structure (e) (II)

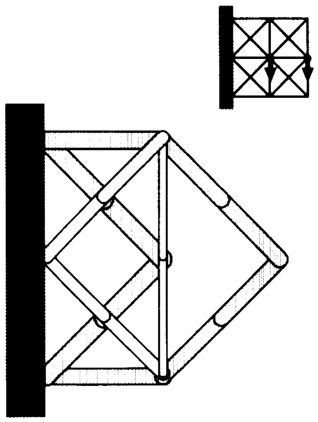

Fig. 10 Optimum structure obtained from ground structure (d)(III)

the ground structure (d). The feature of this truss quite different from the ones shown in Figs. 4, 6 and 7 is the existence of lightly matted members which can be eliminated. The optimum structure generated from (d) is not unique. Other optimum structure of (d) can be obtained as follows. First, the sum of squared values of mass of each member in Fig. 8 is minimized to find a symmetric structure. Secondly, members are removed in the order from the thinnest to the thickest member as many as possible. The structure found by this method is shown in Fig. 9. It is confirmed that the structure in Fig. 9 has as much mass as that in Fig. 8 and satisfies the constraint that the strain energy density is uniform and equal to the prescribed value U. It is anticipated that the contour of the optimum structure based on (d) is decided uniquely, while its inside arrangement of the members is not unique.

Figure 10 shows also the optimum structure of (d). This truss carries two loads as indicated in the small figure, top right of Fig. 10 shows. This figure is evidence that the proposed formula is valid also in the case of more than one load. This truss does not have as many lightly matted members as the structure in Fig. 8 and is heavier than that in Fig. 8.

The optimum structure based on the ground structure (e) shown in Fig. 11 also has removable lightly matted members. A structure shown in Fig. 12 is obtained by elimination of lightly matted members from the structure in Fig. 11 in the same way to obtain the structure in Fig. 9. These figures imply that the optimum structure of (e) is not unique except its contour like that of (d).

The finite element analysis carried out for the structures reveals that the difference between U and the strain energy density in all the optimum structures found in this paper is held less than $4/10^6$.

6. Conclusion

Design parameters to minimize mass of trusses by use of boundary cycle was formulated. The advantages of the proposed formulation are summarized as follows.

1. Useless members which carry no internal force are never generated, because topology of a structure except members with a loading or supporting point is represented as a set of loops.
2. The proposed design parameters can always satisfy the force equilibrium.
3. The connection between a loading point and supporting points is assured.
4. Owing to relationship of mass of each member with its internal force, a constraint condition of uniform distribution of strain energy density and equality to a prescribed value can be easily formulated.

The numerical examples dealt with the minimization of the mass of trusses sustaining a load distant from a rigid wall to examine the validity of the proposed formula. The followings were found.

1. Ground structures with a larger number of members are not always more effective than those with fewer members for the optimization of trusses.
2. In some cases, the optimum structure cannot be determined uniquely by a constraint condition concerning only strain energy density.

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