SYNTHESIS OF LINEAR DISCRETE MULTIDIMENSIONAL CONTROL SYSTEMS

JANUARY 1993

DOCTOR OF ENGINEERING

LI XU

TOYOHASHI UNIVERSITY OF TECHNOLOGY

SYNTHESIS OF LINEAR DISCRETE MULTIDIMENSIONAL CONTROL SYSTEMS

January, 1993

DOCTOR OF ENGINEERING

LI XU

TOYOHASHI UNIVERSITY OF TECHNOLOGY

線形離散 nD 制御系の設計に関する研究

(邦 文 要 旨)

nD（多次元）システムは様々な応用分野が検討され始め, 近年急速に注目されるようになってきた. しかし nD ($n \geq 2$) システムは 1D システムとは全く異る性質があり, 理論上まだ未解決な問題が多く存在する. 本研究ではこれらの未解決な問題のうち, 主としてつぎの 2 つの理論的問題を解明した. 第 1 は, 行列分解表現代数法と Gröbner 基底という数学手法による nD ($n = 2$) システムの安定化, トラッキング及びレギュレータなどの基本制御問題の解法を明らかにした. 第 2 は, Practical な視点より, nD ($n \geq 2$) システムの内部安定性, 可制御性, 可観測性およびフィードバック制御系設計などの問題を論じるとともに, 代数法と状態空間法の両者からそれらの諸問題の理論解明を行なった. これらの内容に関し, 本論文は以下に述べる全 8 章より構成される.

第 1 章は緒論である. 第 2 章と第 3 章は, それぞれ nD システムの解析に必要な基本的な数学知識, nD システムの数学モデルと安定性に関する基本的な概念と結果をまとめたものである.

第 4 章では, 2D システムの安定化問題について論ずる. まず 2D システムの可安定性の判別, 安定な閉ループ多項式の構成法及びΩ-coprime の概念の導入とその条件を明らかにした. ついで安定化問題の解法に多項式 module に対する Gröbner 基底法を持ち込むことにより, 従来の解法が持つ問題点を解決できる新しいアルゴリズムを提案した. 2D システムの安定化問題の解析に, 多項式 module に対する Gröbner 基底の概念を導入したのは本研究が初めてである. これにより安定化問題の基幹をなすユニラテラル多項式行列方程式は, 従来の行列からスカラーへ, さらにスカラーから行列へと煩雑な変換処理を経ずに求解できるようになった.

第 5 章では, 2D システムの deadbeat と漸近サーボ問題を取り上げた. ここでは両者を統一的に取り扱うために, 2D causal Ω-安定な（不安定な領域Ωに極がない）有理関数環上で (zero) skew Ω-prime の概念を導入するとともに, skew Ω-prime 方程式が可解であるための必要十分条件とその解法を与えた. さらにバイラテラル 2D 多項式行列方程式についても, その可解条件と解法を明らかにした. 以上の結果をもとに, deadbeat, 漸近トラッキングおよびレギュレータ問題は skew Ω-prime 方程式を含む数種の代数方程式に帰着され統一的に

解決できることを示した. なお 2D システムの漸近サーボ問題は本研究で初めて完全に解決されたといえる.

画像信号, 地震データ, また 2D ダイナミックス特性を持つ学習制御系, multipass プロセスなどのような実際の nD 信号とシステムの多くは, n 個の独立変数の中で, 一つだけが無限になることができ, それ以外はすべて有限変域区間に制限されているといった特徴がある. Agathoklis 氏らは, nD システムのこの特徴を考慮して, Practical-BIBO 安定性の概念を導入した. さらに, Practical-BIBO 安定判別は 1D 安定性問題に帰着できることを示し, 従来の BIBO 安定性は多くの実際の nD システムには厳しすぎることを明らかにした. 第 6 章は, この Practical-BIBO 安定の概念をもとに, 代数法による nD システムの Practical 安定化問題について考察し, 従来の BIBO 安定化補償器の設計法は適用できないことを指摘し, 新たな設計法を構築した.

第 7 章では, Roesser 型 nD 状態変数モデルに基づく Practical 内部 (漸近) 安定の概念を導入し, その必要十分条件を示した. また第 6 章と同様 Practical 漸近安定性も n 個の 1D システムの安定性と等価であり, 従来の (2D) 内部安定性の定義も実問題において厳しすぎることを明らかにした. さらに Practical な意味での可制御, 可観測を考察し, 状態フィードバックによる Practical 安定化と Practical な漸近状態推定オブザーバの構成についても検討した. これらの考察をとおして, nD システムの Practical 内部安定性と Practical-BIBO 安定性, および Practical 安定化補償器の設計に対する代数法と状態空間法との関係を明らかにすることができた.

第 6 章と第 7 章の結果より, Practical な意味での nD 制御問題は, すべて 1D の同類問題に帰着でき, 結果的に 1D の技法で解決できるといった nD システム制御理論の実用化にとって重要な性質が明らかになった.

第 8 章は結言で, 本論文の主要な結果をまとめるとともに, 第 4 章の結果の nD ($n > 2$) への拡張の可能性, 第 6, 7 章の Practical 安定なフィードバック補償器の設計技法を学習制御系, multipass プロセスへ適用する可能性など今後の研究課題について述べた.

Synthesis of Linear Discrete Multidimensional Control Systems

(Abstract)

This thesis considers the synthesis problem of nD (n-dimensional) feedback control systems. The two-fold objective of the thesis is to systematically treat

- the output feedback stabilization and servo problems for 2D systems by means of MFD (matrix fractional description) algebraic and Gröbner basis approaches; and
- some fundamental properties and feedback control problems for nD systems in a practical sense, i.e., under the assumption that the input and output signals of the systems are unbounded in, at most, one dimension, by MFD algebraic approach as well as state-space approach.

Corresponding to this objective, the main contents and results are concerned with the topics:

1. Output feedback stabilizability and stabilization algorithms for 2D systems
2. Skew Ω-primeness and servo problems for 2D systems
3. Design of practically-stable nD feedback systems

and the thesis is organized as follows:

Chapter 1 is a general introduction of the background, motivations and so on for the research. With a view to making the thesis self-contained, some mathematical preliminaries and fundamental results on description and stability of nD systems, central to the developments which follows, are briefly reviewed in Chapter 2 and Chapter 3, respectively. Chapter 4 is devoted to a detailed study on the stabilizability and stabilization of 2D systems. In particular, we propose some alternative methods for test of output feedback stabilizability and construction of a stable 2D closed-loop polynomial. Also, we prove a generalization of the well-known "Rabinowitsch trick" in some sense to the case of modules over polynomial ring which allows a possibility to solve the stabilization problem effectively by using the Gröbner basis approach for modules. Chapter 5 focuses on the solution of various problems concerning 2D skew Ω-primeness and 2D bilateral polynomial matrix equations. Based on these results, then, the 2D asymptotic and deadbeat tracking and disturbance rejection problems are solved in a unified way. Chapter 6 is addressed to the problem of practical-stabilization of nD systems by MFD algebraic approach. In Chapter 7, some fundamental properties and control problems of nD systems in the practical sense are considered from state-space point of view. Based on nD Roesser state-space model, the concept of practical internal stability is introduced and a necessary and sufficient condition is derived. Then, the notions of practical-controllability, practical-observability, etc. are presented and associated conditions are shown. Meanwhile, the problems of practical-stabilization by local state feedback and construction of state observer in the practical sense are solved. Further, the relationship between practical-BIBO stability and practical internal stability, and a connection between the algebraic and state-space approaches will be clarified as well. Finally, Chapter 8 gives a brief summary of the main results of the thesis and some remarks on further possibilities and problems for future research.

Contents

List of Figures

Chapter 1

Introduction

1.1 Background and Objectives of the Research

Over the last two decades, growing interests and considerable contributions have been developed on the theory of multidimensional (nD) systems, namely, the systems that are characterizable by more than one independent variables. This is, of course, mainly due to the diversity of the actual and potential applications of nD systems embracing multidimensional signal processing, variable-parameter and lumped-distributed network synthesis, computer memory, delay-differential systems, iterative learning control systems, linear multipass processes, etc. (see, e.g., [19–21, 29, 100] and the references therein). On the other side, this is also undoubtedly due to the fact that nD system theory has exposed an entirely new research area whose theoretic interest largely exceeds their immediate application. From the beginning, as a matter of fact, there exist some deep and substantial differences between the nD ($n > 1$) system theory and the conventional 1D system theory such that the standard 1D concepts and techniques are no longer sufficient for nD situations. In order to analyze and synthesize nD dynamic systems, therefore, some entirely new notions and techniques have to be introduced.

Some fundamental difficulties for nD systems arise from the more complex algebraic structure exhibited by the ring of multivariate polynomials that provides a ground setting for description and study of a large class of nD systems. Unlike 1D polynomial ring, first of all, the ring of nD ($n > 1$) polynomials is not an Euclidean ring, thus most 1D techniques that involve Euclidean division, such as elementary operations of matrices, are not applicable (see, e.g., [20, 93]). Moreover, there may possibly exist common zeros in two multivariate polynomials even when they share no common factor, which is impossible for

1D polynomials. These zeros are referred to as nonessential singularities of the second kind if the two polynomials are considered as the numerator and the denominator of an nD rational function [20]. In the case of $n = 2$, these singularities occur as isolated points, while in the cases of $n > 2$ these singularities are usually surfaces or manifolds. For these features of nD polynomials, the notion of coprimeness might have several nonequivalent interpretations. In particular, for nD polynomials and/or nD polynomial matrices, the concepts of factor coprimeness, zero coprimeness and minor coprimeness have been recognized [123]. There are different corresponding relations between these definitions of coprimeness according to different values of n in nD polynomial matrices. For $n = 1$, all the three definitions are equivalent each other. For $n = 2$, the factor coprimeness and the minor coprimeness are equivalent but they do not imply the zero coprimeness. For $n \geq 3$, there exists no equivalence between any pair of the definitions. Always, however, the following relation holds true: zero coprimeness ⇒ minor coprimeness ⇒ factor coprimeness. These characteristic properties make the control problems of nD systems, such as analysis of stability, synthesis of feedback control and so on, much more complicated and difficult than the corresponding 1D problems.

As in the 1D case, the study of various problems of nD systems may be carried out by either state space or algebraic approaches. In this research, particular attention will be payed to algebraic approach for the following reasons:

First, recent research results (see, e.g., [62, 103]) have proven that the algebraic approach based on the "coprime" factorization of system transfer function matrices over a specified ring, which is also often referred to simply as *factorization approach* or *matrix fractional description* (*MFD*) *approach*, provides a natural and elegant means to the solution of a variety of important control problems. Another attractive feature of this approach is that it generally encompasses continuous-time as well as discrete-time systems, lumped as well as distributed systems, and 1D as well as nD ($n > 1$) systems, all within a single framework. The principal idea is to reduce the synthesis procedure to solving certain linear (matrix) equations over the ring of polynomials or (proper) stable rational functions. In particular, it is well known that the problems of characteristic polynomial assignment and stabilization for an MIMO (multi-input multi-output) system can be formulated to

the solution of unilateral matrix equation over the polynomial ring [62], i.e., the equation

$$DX + NY = \Phi \qquad (\text{or} \quad XD + YN = \Phi) \tag{1.1}$$

where D, N are polynomial matrices specified by the corresponding plant and X, Y are polynomial matrices to be found, and in particular Φ is a stable polynomial matrix, given for characteristic polynomial assignment problem and to be constructed for stabilization problem. Equivalently, the stabilization problem can be also formulated to the solution of Bezout equation over the ring of stable rational functions [103], i.e., the equation

$$DX + NY = I \qquad (\text{or} \quad XD + YN = I) \tag{1.2}$$

where D, N, X,Y are now all stable rational functions and I is the identity matrix. Further, the problem of tracking control can be characterized by bilateral equation [62], i.e.,

$$DX + YN = \Phi \tag{1.3}$$

where D, N, Φ are given polynomial matrices, and X, Y are unknown polynomial matrices; or skew prime equation [103], i.e.,

$$DX + YN = I \tag{1.4}$$

where D, N, X, Y are polynomial matrices or stable rational matrices corresponding to the deadbeat or asymptotic tracking problems, respectively. For the 1D case, the mentioned linear matrix equations are now well understood (see, e.g., [62, 103, 108]). For the nD cases, however, a lot of work remains to be done.

Secondly, the basic nD problems and techniques concerning state-space approach have been studied in rather detail by some researchers (see, e.g., [9, 15, 38, 40, 91, 101] and the references therein). Interesting enough, it is observed that some nD control problems in state-space, such as stabilization by local state or output feedback and deadbeat control, are actually solved by formularizing them back to certain linear matrix equations described in transfer function domain. This is because that in the nD case, dynamic state feedback has in general more potentiality than the static one [9], while it is well known in the 1D case both static and dynamic feedback have the same potentiality. This fact means that in nD control systems there exists often the need of dynamic feedback rather than the static one, however it is difficult to construct a design procedure for such purpose in the state-space. In some sense, therefore, algebraic approach in fact plays a quite fundamental role in nD system theory.

Thirdly, as mentioned previously, nD systems possess very close connection with the theory of nD polynomials and nD polynomial matrices, and effective means are naturally required above all for tackling the computations concerned with nD polynomials and nD polynomial matrices. For this purpose, Gröbner basis is rather attractive among various methods. Gröbner basis was introduced by Buchberger [23], and in the area of computer algebra it is now a well-known powerful method for treatment of nD polynomial computations. By this method quite a few mathematical problems concerned with nD polynomial ideals have been successfully solved (see, e.g, [23] and the references therein). It is expected, therefore, that this method will also be very useful in nD system theory, and in fact some favorable developments have been achieved [12, 49, 85–88].

According to such background, one of the main objectives of this research is to investigate, in a systematical and unified way, the possibility to solve fundamental nD control problems by using MFD algebraic approach and Gröbner basis approach. Since the stability analysis of nD systems, as a most fundamental start point, has been investigated quite extensively and intensively in the literature (see, e.g., [5, 6, 35, 45, 50, 51, 96, 98]), attention of this research will be mainly concentrated to the problems of feedback stabilization and tracking control of 2D systems.

On the other hand, if we consider some practical situation of nD signal processing, for example, seismic, sonar and television image processing, we find that these systems possess the following common feature. Namely, in practical situations, the independent variables $i_1, \cdots, i_n$ of an nD signal $x(i_1, \ldots, i_n)$ are usually spatial variables, except that perhaps one of the variables, say i_j, is a temporal variable. Actually, the temporal variable is unbounded whereas the other spatial variables are almost always bounded. Taking this feature into account, Agathoklis and Bruton [1] have developed the concept of *practical-BIBO stability* for nD discrete systems, and shown that the analysis of practical-BIBO stability can be equivalently reduced to the 1D stability problem of n 1D systems. In fact, this shows that the conventional-BIBO stability are too restrictive for many practical applications. As mentioned earlier, it has been reported that iterative learning control systems and linear multipass process can be described by 2D system model, and further

analyzed by using 2D system theory [16, 44, 72, 83]. As a common feature of these systems, we again find that while the iterations are not subjected to any boundary condition, the iterated passes are usually bounded on a finite discrete-time interval. There is therefore significant need to consider the control problems of such systems under the concept of practical-BIBO stability. Up to now, some results have been documented in the literature (e.g. [80]) for the design of practical-BIBO stable nD digital filters. However, for practical-stabilization of nD control systems by feedback scheme, a general effective method has not yet been available.

Another objective of this research, then, is to resolve various nD control problems in the practical sense of [1] in the hope that the research results might be helpful to widen the way for practical applications of nD system control theory.

1.2 Main Results and Arrangement of the Thesis

Corresponding to the objectives stated above, the main contents and results of the thesis are concerned with the following aspects:

I. Output feedback stabilizability and stabilization algorithms for 2D systems

Alternative methods for test of output feedback stabilizability and construction of a stable closed-loop polynomial for 2D systems are proposed. By these methods, the problems can be reduced to the 1D case and solved by using 1D algorithms. Another feature of the proposed methods is that their extension to certain special nD ($n > 2$) cases can be readily obtained.

Then, the "Rabinowitsch trick", a technique ever used in showing the well-known Hilbert's Nullstellensatz, is generalized in some sense to the case of modules over polynomial ring. These results eventually lead to two new constructive solution algorithms for the 2D unilateral polynomial matrix equation $D(v,w)X(v,w) + N(v,w)Y(v,w) = \Phi(v,w)$ with $\Phi(v,w)$ being stable, which arises in the 2D feedback design problem. Although one of the algorithms does not appear to provide particular advantage for practical computation, it reveals some theoretic insights and plays an essential role in the proofs of the above results. Complementally, the other one shows that the equation can be effectively solved by transforming it to an equivalent Bezout equation so that the Gröbner basis approach for modules over polynomial ring can be directly applied. As an apparent feature, this method removed several questions but shares the advantages of existing algorithms (see Chapter 4 for detailed discussions).

Moreover, the concepts of factor coprimeness and zero coprimeness in the 2D polynomial ring are generalized to the ring of causal Ω-stable 2D rational functions as factor Ω-coprimeness and zero Ω-coprimeness, respectively. Here, Ω is a specified subdomain of $\mathbf{C}^2$ containing the origin, and a 2D rational functions is Ω-stable if it has no poles in Ω. A necessary and sufficient condition for the existence of zero Ω-coprime MFD is then derived by using the results for 2D unilateral matrix equation. Based on these results, the problems of stabilization of 2D systems and parametrization of all stabilizing compensators can be easily solved in the standard way [49, 103].

II. Skew Ω-primeness and servo problems for 2D systems

Motivated by the asymptotic and deadbeat servo control problems in 2D system theory, two special cases of the bilateral 2D polynomial matrix equation $DX + YN = \Phi$ when $\Phi = I$ and $\Phi = \phi I$ with ϕ a Ω-stable 2D polynomial are first considered. For the sake of conceptual and methodological brevity, we attack the two problems in a unified way. Namely, instead of using the ring of 2D polynomials, we consider the problems over the more general ring of Ω-stable 2D rational functions which contains the 2D polynomial ring as a subring. The concept of zero skew primness [94] in the 2D polynomial ring is generalized to the ring of causal Ω-stable 2D rational functions as (zero) skew Ω-primness. Thus, the two problems above can be unitedly formulated to the skew Ω-prime equation $DU + VN = I$, where D, N are 2D polynomial matrices and U, V are matrices whose entries belong to the ring of Ω-stable 2D rational functions. Necessary and sufficient conditions for skew Ω-primeness are derived. Then, a constructive solution procedure for the skew Ω-prime equation is proposed by employing several 2D results developed in [12, 48, 74]. The proposed procedure applies under a less restrictive condition than the known one [94] when the equation is considered over the ring of 2D polynomials.

Based on the above results, we derive a solvability condition for the bilateral 2D polynomial matrix equation when Φ is a general 2D polynomial matrix. The general solutions are investigated as well.

Further, the uniqueness of the skew complement pair of a pair of skew Ω-prime matrices

with respect to $\mathbf{H}$-unimodular equivalence is considered and some associated conditions are derived. A counterexample is also given for Emre's conjecture [33] on the "fixed poles" of the skew prime equation over the 1D polynomial ring.

It should be remarked that most results stated above follow very closely those developed by Wolovich [108] for 1D skew prime polynomial matrices. However, the extension of these results to the 2D case is far from trivial.

By making use of the results for 2D Ω-coprime and skew Ω-prime equations, the asymptotic and deadbeat output regulation and tracking problems of 2D systems are solved in a unified way. Necessary and sufficient conditions for the problems are first given generally in terms of 2D matrix equations over the ring of Ω-stable rational functions. Then, depending on the selection of $\Omega = \bar{U}^2$ or $\Omega = \mathbf{C}^2$, we can get the corresponding asymptotic or deadbeat solutions for the problems under consideration, respectively. For, the ring of Ω-stable rational functions reduces in fact to the polynomial ring when $\Omega = \mathbf{C}^2$.

III. Design of practically-stable nD feedback systems

Based on the concept of practical-BIBO stability introduced by Agathoklis and Bruton [1], several fundamental control problems are investigated for nD systems whose input and output signals are unbounded in, at most, one dimension.

First of all, by using algebraic approach, the feedback practical-stabilization problem of nD systems is considered and solved. A constructive algorithm is proposed for solving Bezout equation over the ring of practically-stable rational functions. A necessary and sufficient condition for an nD system to be practically-stabilizable is derived and the class of all nD practically-stabilizing compensators is parametrized. The proposed method and some basic properties of practically-stable systems are illustrated by some examples.

Then, the concept of practical internal stability, or practical asymptotic stability, is introduced based on nD Roesser state-space model, and a necessary and sufficient condition for the stability is given. This condition reveals that practical internal stability is equivalent to the stabilities of n 1D systems. This fact implies that the conventional definition of internal stability for 2D system [2, 40, 41] is also unnecessarily restrictive for many practical applications, just as that shown by Agathoklis and Bruton [1] for practical-BIBO stability and conventional-BIBO stability.

Next, the notions of practical-controllability and practical-observability are established and associated necessary and sufficient conditions are derived. We also show the solvability of the problems of practical-stabilization by local state feedback and construction of asymptotic state observer in the practical sense. Based on these results, the relationship between practical-BIBO stability and internal practical stability is clarified under the concepts of practical-detectability and practically-stabilizability.

Similarly to the 1D case [77], a connection between state-space and doubly coprime MFD on the ring of practically-stable rational functions can be given.

The obtained results make it clear that all the above-mentioned control problems in the practical sense of [1] can be formulated to the corresponding 1D problems, therefore can be essentially resolved by using 1D methods.

The thesis is organized as follows. With a view to making the thesis self-contained, some mathematical preliminaries and fundamental results on description and stability of nD systems, central to the developments which follows, are briefly reviewed in Chapter 2 and Chapter 3, respectively. Chapter 4 is devoted to a detailed study on the stabilizability and stabilization of 2D systems. In particular, we propose some alternative methods for test of output feedback stabilizability and construction of a stable 2D closed-loop polynomial. Also, we prove a generalization of the well-known "Rabinowitsch trick" in some sense to the case of modules over polynomial ring which allows a possibility to solve the stabilization problem effectively by using the Gröbner basis approach for modules. Chapter 5 focuses on the solution of various problems concerning 2D skew Ω-primeness and 2D bilateral polynomial matrix equations. Based on these results, then, the 2D asymptotic and deadbeat tracking and disturbance rejection problems are solved in a unified way. Chapter 6 is addressed to the problem of practical-stabilization of nD systems by MFD algebraic approach. In Chapter 7, some fundamental properties and control problems of nD systems in the practical sense of [1] are considered from state-space point of view. Based on nD Roesser state-space model, the concept of practical internal stability is introduced and a necessary and sufficient condition is derived. Then, the notions of practical-controllability, practical-observability, etc. are presented and associated conditions are shown. Meanwhile, the problems of practical-stabilization by local state feedback and construction of state observer in the practical sense are solved. Further, the relationship between practical-BIBO stability and practical internal stability, and a connection

between the algebraic and state-space approaches will be clarified as well. Finally, Chapter 8 gives a brief summary of the main results of the thesis and some remarks on further possibilities and problems for future research.

1.3 Notations and Abbreviations

$\mathbf{R}$	the field of real numbers
$\mathbf{C}$	the field of complex numbers
Z	set of integers: $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$
Z_+	set of positive integers: $\{0, 1, 2, \cdots\}$
Z_+^n	$\{(i_1, i_2, \ldots, i_n) \mid i_1, i_2, \ldots, i_n \in Z_+\}$
Z_+^{-n}	$\{(i_1, i_2, \ldots, i_n) \mid i_1, i_2, \ldots, i_n \in Z_+$, but not more than one of them can be infinite simultaneously $\}$
$\bar{U}$	closed unit disc in $\mathbf{C}$, i.e., $\{v \in \mathbf{C} \mid \lvert v\rvert \leq 1\}$
$\bar{U}^n$	closed unit polydisc, i.e., $\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \lvert z_k\rvert \leq 1,\ k = 1, \ldots, n\}$
U^n	open unit polydisc, i.e., $\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \lvert z_k\rvert < 1,\ k = 1, \ldots, n\}$
T^n	$\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \lvert z_k\rvert = 1,\ k = 1, \ldots, n\}$
Ω	a *unstable* subdomain in $\mathbf{C}^2$ containing the origin
$\mathbf{R}[z_1, \ldots, z_n]$	commutative ring of nD polynomials in $z_1, \ldots, z_n$ with coefficients in $\mathbf{R}$
$\mathbf{R}[z_1, \ldots, z_{n-1}][z_n]$	ring of polynomials in z_n with coefficients in $\mathbf{R}[z_1, \ldots, z_{n-1}]$
$\mathbf{R}(z_1, \ldots, z_n)$	commutative ring of nD rational functions in $z_1, \ldots, z_n$ with coefficients in $\mathbf{R}$
$\mathbf{R}^q[z_1, \ldots, z_n]$	module of ordered q-tuples in $\mathbf{R}[z_1, \ldots, z_n]$
$\mathbf{R}^{q\times m}[z_1, \ldots, z_n]$	set of $q \times m$ matrices with entries in $\mathbf{R}[z_1, \ldots, z_n]$

G	ring of nD causal rational functions, i.e., $\{n/d \mid n, d \in \mathbf{R}[z_1, \ldots, z_n], d(0, \ldots, 0) \neq 0\}$
H	Ω-stable rational ring, i.e., the ring consisting of the elements of **G** which have no poles in Ω
I	the set of elements of **H** which are units of **G**, i.e., $\{h \in \mathbf{H} \mid h^{-1} \in \mathbf{G}\}$
J	the subgroup of **H** consisting of all invertible elements of **H**, i.e., $\{h \in \mathbf{H} \mid h^{-1} \in \mathbf{H}\}$
$\tilde{\mathbf{H}}$	ring of practically-stable rational functions, i.e., $\{n/d \mid n, s \in \mathbf{R}[z_1, \ldots, z_n], d(0, \ldots, z_k, \ldots, 0) \neq 0, \forall z_k \in \bar{U}, k = 1, \ldots, n\}$
$\tilde{\mathbf{I}}$	$\{h \in \tilde{\mathbf{H}} \mid h^{-1} \in \mathbf{G}\}$
$\tilde{\mathbf{J}}$	$\{h \in \tilde{\mathbf{H}} \mid h^{-1} \in \tilde{\mathbf{H}}\}$
$\mathbf{M}(*)$	set of matrices with appropriate dimensions with entries in $*$, e.g., $\mathbf{M}(\mathbf{R}[z_1, \ldots, z_n])$, $\mathbf{M}(\mathbf{G})$, $\mathbf{M}(\mathbf{H})$, etc.
$\mathcal{I}$	ideal generated by, e.g., $\{f_1, \ldots, f_m \mid f_i \in \mathbf{R}[z_1, \ldots, z_n],\ i = 1, \ldots, m\}$, i.e, $\{f_1h_1 + \cdots + f_mh_m \mid h_i \in \mathbf{R}[z_1, \ldots, z_n],\ i = 1, \ldots, m\}$
$\mathcal{V}(\mathcal{I})$	the algebraic variety of $\mathcal{I}$, i.e., the set of the common zeros of $f_1, \ldots, f_m$
$\det A$	determinant of matrix A
rank A	rank of matrix A
A^T	transpose of matrix A
A^{-1}	inverse of matrix A
$\lvert \cdot \rvert$	the absolute value of a real or complex number
$\lVert \cdot \rVert$	a norm, e.g., Euclidean norm of a vector in $\mathbf{R}^n$ or $\mathbf{C}^n$
Re v	the real part of a complex v
max	maximum function

min	minimum function
sgn	"signum" function: sgn x is 1, 0, or -1, respectively, according as $x > 0$, $x = 0$, $x < 0$
$\forall$	"for all"
$\exists$	"there exist(s)"
$\triangleq$	"is defined by"
$a \in A$	a is an element of A; a belongs to A
$A \subset B$	A is a subset of B
$A \cup B$	union of set A and set B
$A \cap B$	intersection of set A and set B
$a \Rightarrow b$	a implies b; equivalently, "not a" implies "not b"
$a \Leftarrow b$	b implies a
$a \Leftrightarrow b$	a if and only if b
nD	n-dimensional; multidimensional
MFD	matrix fractional description
BIBO	bounded-input bounded-output
SISO	single-input single-output
MIMO	multi-input multi-output
g.c.f.	greatest common factor
lcm	least common multiple
FR(L)C	factor right (left) coprime
MR(L)C	minor right (left) coprime
ZR(L)C	zero right (left) coprime

Chapter 2

Mathematical Preliminaries

2.1 Multivariate Polynomials, Rational Functions, and Matrices

Let $\mathcal{R}$ be a ring with identity, n a positive integer. Then, by *a polynomial* $a(x_1, \ldots, x_n)$ *over* $\mathcal{R}$ *in* n *indeterminates* $x_1, \ldots, x_n$, we mean a finite sum of the form

$$a(x_1, \ldots, x_n) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} a_{k_1,\ldots,k_n} x_1^{k_1} \cdots x_n^{k_n} \tag{2.1}$$

where $k_1, \ldots, k_n$ are non-negative integers and coefficients $a_{k_1,\ldots,k_n}$ belong to the ground ring $\mathcal{R}$.

The sum of the exponents $k_1 + k_2 + \cdots + k_n$ is called the degree of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, while the maximum d of the degrees of the monomials of a non-zero polynomial $a(x_1, \ldots, x_n)$ is called the degree of $a(x_1, \ldots, x_n)$ with respect to all its variables, or *total degree.* By the degree d_{x_i} of the polynomial $a(x_1, \ldots, x_n)$ with respect to one of the variables x_i $(i = 1, 2, \ldots, n)$, we mean the highest exponent with which x_i occurs in the terms of the polynomial:

$$d_{x_i} = m_i \quad \text{for} \quad i = 1, 2, \ldots, n; \quad d = \sum_{k=1}^{n} m_k \quad a_{m_1 m_2,\cdots m_n} \neq 0. \tag{2.2}$$

If all the monomials in $a(x_1, \ldots, x_n)$ have the same degree d, then $a(x_1, \ldots, x_n)$ is said to be *homogeneous* or to be a *form* [124].

It is easy to see that the set of all polynomials over $\mathcal{R}$ in n indeterminates, denoted by $\mathcal{R}[x_1, \cdots, x_n]$, is a commutative ring. In view of the context of nD discrete systems to be considered in this thesis, unless otherwise specified, from now on the ground ring will be the field of real numbers $\mathbf{R}$ and the indeterminates will be complex indeterminates, represented by $z_1, \ldots, z_n$ (or for brevity v, w when $n = 2$).

Then, $\mathbf{R}^{p\times m}[z_1, \ldots, z_n]$ stands for the set of $p\times m$ matrices with entries in $\mathbf{R}[z_1, \ldots, z_n]$.

It is also easy to note that every element of $\mathbf{R}[z_1, \ldots, z_n]$ can be considered as a polynomial in z_k with coefficients in $\mathbf{R}[z_1, \ldots, z_j, \ldots, z_n]$, $j \neq k$, so $\mathbf{R}[z_1, \ldots, z_n] = \mathbf{R}[z_1, \ldots, z_j, \ldots, z_n][z_k]$ for all k $(= 1, \ldots, n)$.

Furthermore, by $\mathbf{R}(z_1, \ldots, z_n)$ we denote the field of rational functions, namely, the quotient field of $\mathbf{R}[z_1, \ldots, z_n]$. An element $p(z_1, \ldots, z_n)$ of $\mathbf{R}(z_1, \ldots, z_n)$ is a quotient of two n-variate polynomials $a(z_1, \ldots, z_n)$, $b(z_1, \ldots, z_n) \in \mathbf{R}[z_1, \ldots, z_n]$:

$$p(z_1, \ldots, z_n) = \frac{b(z_1, \ldots, z_n)}{a(z_1, \ldots, z_n)} \tag{2.3}$$

with $a(z_1, \ldots, z_n) \not\equiv 0$.

The set of all $p \times m$ n-variate rational matrices with entries in $\mathbf{R}(z_1, \ldots, z_n)$ will be denoted by $\mathbf{R}^{p\times m}(z_1, \ldots, z_n)$.

2.2 Singularities of n-variate Rational Functions

Entirely differing from the univariate case, two n-variate $(n > 1)$ polynomials might have common zeros even if they are devoid of common polynomial factors other than real numbers, i.e., factor coprime.

For a rational function $p(z_1, \ldots, z_n)$ as in Equation (2.3), when $b(z_1, \ldots, z_n)$ and $a(z_1, \ldots, z_n)$ are factor coprime, their common zeros are referred to as *nonessential singularities of the second kind*, while the zeros of $a(z_1, \ldots, z_n)$ that are not simultaneously the zeros of $b(z_1, \ldots, z_n)$ are referred to as *nonessential singularities of the first kind* (see, e.g., [20]).

It has been shown in the literature that nonessential singularities of the second kind are responsible for several major problems in nD system theory [20, 21, 54, 100]. In the case of $n = 2$, these singularities occur as isolated points, and in the cases of $n > 2$, these singularities are usually curves and surfaces.

2.3 Coprimeness and Factorization of Multivariate Polynomial Matrices

A n-variate polynomial matrix $C(z_1, \ldots, z_n)$ (respectively $B(z_1, \ldots, z_n)$) is called a right (left) factor of the matrix $A(z_1, \ldots, z_n)$ if there exists a matrix $B(z_1, \ldots, z_n)$ $(C(z_1, \ldots, z_n))$ such that

$$A(z_1, \ldots, z_n) = B(z_1, \ldots, z_n)C(z_1, \ldots, z_n) \tag{2.4}$$

and then $A(z_1, \ldots, z_n)$ is called a left (right) multiple of $C(z_1, \ldots, z_n)$ $(B(z_1, \ldots, z_n))$.

A n-variate polynomial matrix $U(z_1, \ldots, z_n)$ $(V(z_1, \ldots, z_n))$ is called a common right (left) factor of $A(z_1, \ldots, z_n)$ and $B(z_1, \ldots, z_n)$ if there exist two n-variate polynomial matrices $A_1(z_1, \ldots, z_n)$, $B_1(z_1, \ldots, z_n)$ $(A_2(z_1, \ldots, z_n)$, $B_2(z_1, \ldots, z_n))$ such that

$$A = A_1U, \quad B = B_2U$$

$$(A = VA_2, \quad B = VB_2) \tag{2.5}$$

Moreover, $U(z_1, \ldots, z_n)$ $(V(z_1, \ldots, z_n))$ is called right (left) greatest common factor (g.c.f.) of $A(z_1, \ldots, z_n)$ and $B(z_1, \ldots, z_n)$ if $U(z_1, \ldots, z_n)$ $(V(z_1, \ldots, z_n))$ is, in addition, left (right) multiple of every common right (left) factor of $A(z_1, \ldots, z_n)$ and $B(z_1, \ldots, z_n)$.

Definition 2.1 [20, 123]

Let $A(z_1, \ldots, z_n) \in \mathbf{R}^{r\times r}[z_1, \ldots, z_n]$, $B(z_1, \ldots, z_n) \in \mathbf{R}^{q\times r}[z_1, \ldots, z_n]$, *and let*

$$F(z_1, \ldots, z_n) = \begin{bmatrix} A(z_1, \ldots, z_n) \\ B(z_1, \ldots, z_n) \end{bmatrix} \tag{2.6}$$

Then $A(z_1, \ldots, z_n)$, $B(z_1, \ldots, z_n)$ *are said to be*

(i) *factor right coprime* (FRC) *if for any polynomial matrix decomposition*

$$F(z_1, \ldots, z_n) = F_1(z_1, \ldots, z_n)F_2(z_1, \ldots, z_n) \tag{2.7}$$

the $r \times r$ *matrix* $F_2(z_1, \ldots, z_n)$ *is unimodular, i.e.,* $\det F_2(z_1, \ldots, z_n) \in \mathbf{R}\backslash\{0\}$;

(ii) *minor right coprime* (MRC) *if all the* $r\times r$ *minors of* $F(z_1, \ldots, z_n)$ *are factor coprime;*

(iii) *zero right coprime* (ZRC) *if there exists no* n*-tuple* $(z_1, \ldots, z_n) \in \mathbf{C}^n$ *which is a common zero of all the above minors.*

In a dual manner, $A(z_1,\dots,z_n)$, $B(z_1,\dots,z_n)$ *are zero left coprime* (ZLC), *etc., if* $A^T(z_1,\dots,z_n)$ *and* $B^T(z_1,\dots,z_n)$ *are* ZRC, *etc.*

The following theorem reveals the relations among these definitions of coprimeness in n-variate polynomial matrices when n takes different values. The results are only summarized for right coprimeness.

Theorem 2.1 [123]

For $n = 1$, FRC $\equiv$ MRC $\equiv$ ZRC; *for* $n = 2$, FRC $\equiv$ MRC $\not\equiv$ ZRC; *and for* $n \geq 3$, FRC $\not\equiv$ MRC $\not\equiv$ ZRC. *However, for all* $n \geq 1$, ZRC $\Rightarrow$ MRC $\Rightarrow$ FRC.

In nD system theory problems, there is often the need to test the factor coprimeness of two polynomials (or polynomial matrices) which may be given as the numerator (matrix) and denominator (matrix) of a system transfer function (matrix), respectively. As for the n-variate polynomial case, tests proposed in [17, 21, 34] can be applied. When the polynomials are not factor coprime, algorithms for extraction of their g.c.f. can be found in [18, 20]. For g.c.f. extraction of multivariate polynomial matrices, the algorithms proposed by Morf *et al.* [74] and Guiver and Bose [48] are applicable for the case of $n = 2$. Since these algorithms are based on the primitive factorization, and Youla and Gnavi [123] have demonstrated the infeasibility in general of primitive factorization for polynomial matrices in three variables via a counter example, these algorithms fail in the case of $n > 2$. Lai and Chen [67] recently proposed a new procedure not depending on primitive factorization which is possibly applicable to $n \geq 2$ cases. For brevity, these results are not reviewed here.

2.4 Factor Coprime Matrix Fractional Description (MFD) of Multivariate Polynomial Matrices

Consider the problem of factoring n-variate rational matrix $P(z_1,\dots,z_n) \in \mathbf{R}^{q\times r}(z_1,\dots,z_n)$ in the form

$$P(z_1,\dots,z_n) = B(z_1,\dots,z_n)A^{-1}(z_1,\dots,z_n) \tag{2.8}$$

where $A(z_1,\dots,z_n) \in \mathbf{R}^{r\times r}[z_1,\dots,z_n]$, $B(z_1,\dots,z_n) \in \mathbf{R}^{q\times r}[z_1,\dots,z_n]$ are FRC.

The desired factorization can be performed as follows. First, it is always possible to write the $q \times r$ matrix $P(z_1,\dots,z_n)$ in the form

$$\begin{aligned} P(z_1,\dots,z_n) &= \begin{bmatrix} n_{11}/d_1 & n_{12}/d_2 & \cdots & n_{1r}/d_r \\ \vdots & \vdots & & \vdots \\ n_{q1}/d_1 & n_{q2}/d_2 & \cdots & n_{qr}/d_r \end{bmatrix} \\ &= \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1r} \\ \vdots & \vdots & & \vdots \\ n_{q1} & n_{q2} & \cdots & n_{qr} \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_r \end{bmatrix}^{-1} \\ &\triangleq \tilde{B}(z_1,\dots,z_n)\tilde{A}^{-1}(z_1,\dots,z_n), \quad d_k \not\equiv 0, k = 1,\dots,r \end{aligned} \tag{2.9}$$

Then by using the algorithm of [67], a right greatest common factor $D(z_1,\dots,z_n)$ of $\tilde{A}(z_1,\dots,z_n)$ and $\tilde{B}(z_1,\dots,z_n)$ can be extracted and thus $A(z_1,\dots,z_n)$ and $B(z_1,\dots,z_n)$ which are factor right coprime can be obtained from $A = \tilde{A}(z_1,\dots,z_n)\cdot D^{-1}(z_1,\dots,z_n)$ and $B = \tilde{B}(z_1,\dots,z_n)D^{-1}(z_1,\dots,z_n)$. As mentioned previously, for the case of $n = 2$, the algorithms of [48, 74] are also available.

2.5 Some Fundamental Properties on Polynomial Matrices and MFD

In this section, we summarize some fundamental properties on multivariate polynomial matrices and MFD's, which will be applied later.

Theorem 2.2 [20, 123]

The $q\times r$ *and* $m\times r$ *polynomial matrices* $A(z_1,\dots,z_n)$, $B(z_1,\dots,z_n)$ *with* $q+m \geq r \geq 1$ *are ZRC if and only if there exist polynomial matrices* $X(z_1,\dots,z_n)$, $Y(z_1,\dots,z_n)$ *of appropriate dimensions such that*

$$X(z_1,\dots,z_n)A(z_1,\dots,z_n) + Y(z_1,\dots,z_n)B(z_1,\dots,z_n) = I_r \tag{2.10}$$

where I_r *is the* $r \times r$ *identity matrix.*

Theorem 2.3 [20, 123]

The $q\times r$ *and* $m\times r$ *polynomial matrices* $A(z_1,\dots,z_n)$, $B(z_1,\dots,z_n)$ *with* $q+m \geq r \geq 1$ *are MRC if and only if for every* $i = 1 \to n$, *there exist polynomial matrices* $X_i(z_1,\dots,z_n)$,

$Y_i(z_1,\ldots,z_n)$ such that

$$X_i(z_1,\ldots,z_n)A(z_1,\ldots,z_n)+Y_i(z_1,\ldots,z_n)B(z_1,\ldots,z_n)$$
$$=\psi_i(z_1,\ldots,z_j,\ldots,z_n)I_r, \qquad j\neq i \tag{2.11}$$

where $\psi_i(z_1,\ldots,z_j,\ldots,z_n)$, $j\neq i$, is a nontrivial scalar polynomial independent of the variable z_i.

In light of the equivalence of factor coprimeness and minor coprimeness when $n=2$, Theorem 2.3 in fact contains the following result as special case which was first given by Morf *et al.* [74].

Corollary 2.1 [74]

The polynomial matrices $A(z_1,z_2)$ and $B(z_1,z_2)$ are FRC if and only if there exists 1D polynomial matrices $E_1(z_1)$, $E_2(z_2)$, and 2D polynomial matrices $X_1(z_1,z_2)$, $Y_1(z_1,z_2)$ and $X_2(z_1,z_2)$, $Y_2(z_1,z_2)$ such that

$$X_1(z_1,z_2)A(z_1,z_2)+Y_1(z_1,z_2)B(z_1,z_2)=E_1(z_1) \tag{2.12a}$$

$$X_2(z_1,z_2)A(z_1,z_2)+Y_2(z_1,z_2)B(z_1,z_2)=E_2(z_2) \tag{2.12b}$$

Also, from Theorem 2.3 the following results can be obtained.

Corollary 2.2 *Given a rational matrix $P(z_1,\ldots,z_n)\in \mathbf{M}(\mathbf{R}(z_1,\ldots,z_n))$ in the following right and left MFD's:*

$$P(z_1,\ldots,z_n) = B_r(z_1,\ldots,z_n)A_r^{-1}(z_1,\ldots,z_n)$$
$$= A_l^{-1}(z_1,\ldots,z_n)B_l(z_1,\ldots,z_n) \tag{2.13}$$

where $A_r(z_1,\ldots,z_n)$, $B_r(z_1,\ldots,z_n)\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$ are MRC, and $A_l(z_1,\ldots,z_n)$, $B_l(z_1,\ldots,z_n)\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$ are MLC. Then

$$\det A_r(z_1,\ldots,z_n)=\det A_l(z_1,\ldots,z_n) \tag{2.14}$$

Proof. The proof can be given in a similar way as for the 2D case [74]. According to Theorem 2.3, for every $i=1\to n$ there exist polynomial matrices X_{ri}, Y_{ri} and X_{li}, Y_{li}

such that

$$X_{ri}A_r+Y_{ri}B_r=\psi_{ri}I \tag{2.15}$$

$$A_lX_{li}+B_lY_{li}=\psi_{li}I \tag{2.16}$$

where ψ_{ri} and ψ_{li} are nontrivial scalar polynomials independent of the variable z_i. Using these results and noting that $A_lB_r-B_lA_r=0$, we have

$$\begin{bmatrix} X_{ri} & Y_{ri}\\ -B_l & A_l\end{bmatrix}\begin{bmatrix} A_r & -Y_{li}\\ B_r & X_{li}\end{bmatrix}\triangleq UV=\begin{bmatrix}\psi_{ri}I & W\\ 0 & \psi_{li}I\end{bmatrix} \tag{2.17}$$

where U, V are defined in an obvious way and $W=Y_{ri}X_{li}-X_{ri}Y_{li}$. Therefore,

$$\det U\cdot\det V=\det(\psi_{ri}I)\cdot\det(\psi_{li}I)\triangleq\tilde{\psi}_{ri}\cdot\tilde{\psi}_{li} \tag{2.18}$$

which implies that

$$\det U=\alpha_i \tag{2.19}$$

$$\det V=\beta_i \tag{2.20}$$

for some polynomials α_i and β_i independent of the variable z_i. Further, we have

$$U\begin{bmatrix} A_r & 0\\ B_r & I\end{bmatrix}=\begin{bmatrix}\psi_{ri}I & Y_{ri}\\ 0 & A_l\end{bmatrix}, \tag{2.21}$$

hence

$$\det U\cdot\det A_r=\tilde{\psi}_{ri}\cdot\det A_l \tag{2.22}$$

or

$$\alpha_i\det A_r=\tilde{\psi}_{ri}\det A_l. \tag{2.23}$$

Equation (2.23) holds for every $i=1\to n$, and every α_i and ψ_{ri} are independent of the variable z_i. Apparently, this implies that $\det A_r=\det A_l$. □

Corollary 2.3 *Let V, T, $F\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$, and let $P=VT^{-1}F$.*

1) *If V and T are MRC, then P is a polynomial matrix if and only if $T^{-1}F$ is a polynomial matrix;*

2) *If T and F are MLC, then P is a polynomial matrix if and only if VT^{-1} is a polynomial matrix.*

Proof. The sufficiency is obvious, and the proof for the necessity can be found in [123]. For the case of $n = 2$, see also [74]. □

Next, an important result for the 2D case obtained by Bisiacco *et al.* [12, 15] is generalized to nD ($n > 2$) case based on the above results.

Theorem 2.4 *Consider a rational matrix $P \in \mathbf{M}(\mathbf{R}(z_1, \ldots, z_n))$ having the following MFD's*

$$\begin{aligned} P(z_1, \ldots, z_n) &= B_r(z_1, \ldots, z_n)A_r^{-1}(z_1, \ldots, z_n) \\ &= A_l^{-1}(z_1, \ldots, z_n)B_l(z_1, \ldots, z_n) \end{aligned} \tag{2.24}$$

where A_r, $B_r \in \mathbf{M}(\mathbf{R}[z_1, \ldots, z_n])$ are MRC, and A_l, $B_l \in \mathbf{M}(\mathbf{R}[z_1, \ldots, z_n])$ are MLC. Denote by $\mathcal{I}$ and $\bar{\mathcal{I}}$ the ideals generated by the all maximal order minors of the matrices $[A_r^T \ B_r^T]^T$ and $[B_l \ A_l]$, by $\mathcal{V}(\mathcal{I})$ and $\mathcal{V}(\bar{\mathcal{I}})$ the algebraic varieties of $\mathcal{I}$ and $\bar{\mathcal{I}}$, respectively. Then $\mathcal{V}(\mathcal{I})$ is identical to $\mathcal{V}(\bar{\mathcal{I}})$.

Proof. The proof can be shown in the same way of [15]. By the determinantal formula for block matrices, it is immediate to see that

$$\begin{aligned} \det \begin{bmatrix} X & -Y \\ B_l & A_l \end{bmatrix} &= \det A_l \det(X + YA_l^{-1}B_l) \\ &= \frac{\det A_l}{\det A_r} \det(XA_r + YB_r) \end{aligned} \tag{2.25}$$

According to Corollary 2.2, we have that $\det A_r = \det A_l$, hence

$$\det \begin{bmatrix} X & -Y \\ B_l & A_l \end{bmatrix} = \det(XA_r + YB_r). \tag{2.26}$$

Now, assume that $[X \ \ -Y]$ is any permutation of the columns of $[I \ \ 0]$. Then, except for the sign, the right- and left-hand sides of Equation (2.26) are maximal order minors of $[B_l \ A_l]$ and $[A_r^T \ B_r^T]^T$ respectively. Letting $[X - Y]$ vary over the set of all permutations, we obtain a bijective correspondence between the maximal order minors of $[B_l \ A_l]$ and $[A_r^T \ B_r^T]^T$. Therefore, $\mathcal{V}(\mathcal{I}) = \mathcal{V}(\bar{\mathcal{I}})$. □

Finally, we recall the following general factorization theorem for the case $n = 2$.

Theorem 2.5 [74]

Let $A(z_1, z_2)$ be a $r \times r$ matrix with entries in $\mathbf{R}[z_1, z_2]$, and let $\det A = \prod_{i=1}^m a_i(z_1, z_2)$. Then, $A(z_1, z_2)$ can be factored as

$$A(z_1, z_2) = \prod_{i=1}^{m} A_i(z_1, z_2) \tag{2.27}$$

with $\det A_i(z_1, z_2) = a_i(z_1, z_2)$, $i = 1, 2, \ldots, m$.

2.6 Gröbner Basis of Polynomial Ideal

The method of Gröbner basis was introduced by Buchberger [23], and it is a technique that provides algorithmic solutions to a variety of problems connected with ideals generated by finite sets of multivariate polynomials. Some basic concepts and properties of Gröbner basis are briefly reviewed here (see [23] for details).

Let $K[x_1, ..., x_n]$ denote a ring of n-variate polynomials with coefficients over the field K, and let Ideal(F) stand for the ideal generated by $F = \{f_i \in K[x_1, ..., x_n] \mid i = 1, \ldots, m\}$, i.e., the set

$$\mathrm{Ideal}(F) = \{\sum_{i=1}^{m} h_i f_i \mid h_i \in K[x_1, ..., x_n], i = 1, ..., m\} \tag{2.28}$$

Furthermore, f is said to be congruent to g modulo Ideal(F), denoted by $f \equiv_F g$, if $f - g \in \mathrm{Ideal}(F)$.

First, it is necessary to fix an admissible term ordering for the power products. By "admissible" it is meant that the defined term ordering $<_T$ should satisfy at least the following two conditions:

i) $1 <_T t$ for all $t \neq 1$;

ii) if $s <_T t$ then $s \cdot u <_T t \cdot u$

where t, s, u are power products of the form $x_1^{i_1} \cdots x_n^{i_n}$. For example, the *total degree (lexicographic) ordering*, which is $1 <_T x <_T y <_T x^2 <_T xy <_T y^2 <_T x^3 \cdots$ in the case of two variables, or the *purely lexicographic ordering*, which is $1 <_T x <_T x^2 <_T x^3 <_T \cdots <_T y <_T xy <_T x^2y <_T \cdots <_T y^2 <_T xy^2 <_T \cdots$ in the case of two variables, are often used. In fact, these conditions imply that $<_T$ is noetherian, i.e., there exist no infinitely decreasing chains of the form $h_1 >_T h_2 >_T \cdots$, where $h_1 >_T h_2 \Leftrightarrow h_2 <_T h_1$.

With respect to the chosen term ordering $<_T$, the following notations will be used.

cf(f,t) the coefficient of power product t in $f \in K[x_1,...,x_n]$;

lpp(f) the leading product, i.e., the maximal power product with non-zero coefficient in $f \in K[x_1,...,x_n]$ with respect to $<_T$;

lcf(f) the leading coefficient, i.e., the coefficient of the lpp(f).

Definition 2.2 [Reduction]

Let $h \in K[x_1,\ldots,x_n]$. Then h is called a reduction of g modulo $F = \{f_1,\cdots,f_m\}$, denoted by $g \to_F h$, if and only if there exists $f \in F$, $b \in K$ and u a power product such that cf$(g,\ u \cdot \text{lpp}(f)) \neq 0$, $b = $ cf$(g, u \cdot \text{lpp}(f))/\text{lcf}(f)$, and

$$h = g - b \cdot u \cdot f \tag{2.29}$$

Definition 2.3 [Normal form]

h is normal form (or reduced form) modulo F if and only if there is no h' such that $h \to_F h'$.

Then h is a normal form of f modulo F, denoted by NF(F,f), if and only if there is a sequence of reductions

$$f = k_0 \to_F k_1 \to_F k_2 \to_F \cdots \to_F k_q = h \tag{2.30}$$

and h is in normal form modulo F.

Definition 2.4 [S-polynomial]

The S-polynomial corresponding to f_1, f_2 is

$$\text{Sp}(f_1,f_2) = u_1 \cdot f_1 - (c_1/c_2) \cdot u_2 \cdot f_2 \tag{2.31}$$

where $c_i = \text{lcf}(f_i)$, u_i is such that $s_i \cdot u_i =$ the least common multiple of s_1 and s_2, and $s_i = \text{lpp}(f_i)$ for $i = 1,2$.

Definition 2.5 [Gröbner basis]

G is called a Gröbner basis if and only if for all g, h_1, h_2, if h_1 and h_2 are normal form of g modulo G then $h_1 = h_2$.

Most of the algorithmic applications of Gröbner basis are based on the following fundamental properties.

Theorem 2.6 *The following statements are equivalent:*

GP1) *G is a Gröbner basis;*

GP2) *For all f, g: $f \equiv_G g$ if and only if* NF$(G,f) = $ NF(G,g);

GP3) *For all g_1, $g_2 \in G$:* NF$(G, \text{Sp}(g_1,g_2)) = 0$.

Property GP2 implies that if G is Gröbner basis of F (so Ideal(F) =Ideal(G)), then for all $f \in$ Ideal(F), NF$(G,f) = 0$. This property can be easily related to the solvability problem of polynomial equation (see, e.g, [23]).

GP3, indeed, gives a decision algorithm to transform a set $F = \{f_1,\cdots,f_m\}$, which is not a Gröbner basis, into a set $G = \{g_1,\cdots,g_l\}$ such that Ideal(F) =Ideal(G) and G is a Gröbner basis. Such algorithms and more details on the applications of Gröbner basis can be found in such as [23].

2.7 Gröbner Basis of Modules on Polynomial Ring

Furukawa *et al.* [43] and Mora and Möller [73] have independently generalized the concept of Gröbner basis for polynomial ideal to the case of module over polynomial ring, which provides in particular an efficient method to solve a system of linear polynomial equations. A brief summary on the Gröbner basis of module can be given as follows (see also [105]).

Let $F = \{\vec{f}_1,\ldots,\vec{f}_s\}$ be a subset of $K^r[x_1,\ldots,x_n]$. By Module(F) we denote the module generated by F, i.e., the set

$$\text{Module}(F) = \{h_1\vec{f}_1 + \cdots + h_s\vec{f}_s \mid h_i \in K[x_1,\ldots,x_n], i = 1,\ldots,s\}. \tag{2.32}$$

To generalize the notion of *reduction*, we need first to fix an ordering on the r-tuples of power products under certain admissible conditions. In fact we can do this by only fixing the ordering on a subset P of r-tuples of power products which consists of the tuples with only one nonzero component, i.e.,

$$P \triangleq \{(0,\ldots,0,x_1^{i_1}\cdots x_n^{i_n},0,\ldots,0) \mid i_1,\ldots,i_n \in Z_+\} \tag{2.33}$$

The elements of P are called as *power product tuples.* Then a partial ordering $<_M$ on P is defined by

$$(\forall \vec{p}_1,\vec{p}_2 \in P)[\vec{p}_1 <_M \vec{p}_2 \Leftrightarrow ((\exists q \neq 1, q \text{ power product})\ \vec{p}_2 = q \cdot \vec{p}_1)]. \tag{2.34}$$

By an admissible ordering $<_{M(T)}$ on P, we mean any total ordering which satisfies the following properties:

i) $(\forall \vec{p}_1, \vec{p}_2 \in P)[\vec{p}_1 <_M \vec{p}_2 \Rightarrow \vec{p}_1 <_{M(T)} \vec{p}_2]$.

ii) $(\forall \vec{p}_1, \vec{p}_2 \in P)[\vec{p}_1 <_{M(T)} \vec{p}_2 \Rightarrow ((\forall q,\ q \text{ power product})\ q \cdot \vec{p}_1 <_{M(T)} q \cdot \vec{p}_2)]$.

It can be shown that every admissible ordering on P is noetherian [105].

Further, the notations $\leq_M$ and $\leq_{M(T)}$ are defined as $(\vec{p}_1 \leq_M \vec{p}_2) \Leftrightarrow [\vec{p}_1 <_M \vec{p}_2$ or $\vec{p}_1 = \vec{p}_2]$ and $(\vec{p}_1 \leq_{M(T)} \vec{p}_2) \Leftrightarrow [\vec{p}_1 <_{M(T)} \vec{p}_2$ or $\vec{p}_1 = \vec{p}_2]$ for all $\vec{p}_1, \vec{p}_2 \in P$.

Let $<_T$ be an admissible ordering on the power products of $K[x_1, \ldots, x_n]$, for example the purely lexicographic ordering or the total degree (lexicographic) ordering. Let $\vec{p} = (0, \ldots, 0, p_i, 0, \ldots, 0)^T$ and $\vec{q} = (0, \ldots, 0, q_j, 0, \ldots, 0)^T \in P$, where $p_i \neq 0$ occurs at the ith position of $\vec{p}$ and $q_j \neq 0$ at the jth position of $\vec{q}$. The *term first ordering based on* $<_T$ [105], or called *highest-order smallest-suffix ordering* [43], is such an example which determines the ordering $<_{M(T)}$ on P by comparing first p_i and q_j with respect to $<_T$, i.e.,

$$\vec{p} <_{M(T)} \vec{q} \Leftrightarrow [p_i <_T q_j \text{ or } (p_i = q_j \text{ and } i > j)]. \tag{2.35}$$

Another possible one is the *index first ordering based on* $<_T$ which defines $<_{M(T)}$ on P by comparing first the indices i and j, i.e.,

$$\vec{p} <_{M(T)} \vec{q} \Leftrightarrow [i > j \text{ or } (i = j \text{ and } p_i <_T q_j)] \tag{2.36}$$

For example, consider the following elements of $\mathrm{Z}^2[x, y]$ where Z is the domain of integers, and choose the total degree (lexicographic) ordering $(x <_T y)$ as term ordering on $\mathrm{Z}[x, y]$. Then by the term first ordering based on $<_T$ we have

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} <_{M(T)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ x \end{pmatrix} <_{M(T)} \begin{pmatrix} x \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ y \end{pmatrix} <_{M(T)} \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$<_{M(T)} \begin{pmatrix} 0 \\ x^2 \end{pmatrix} <_{M(T)} \begin{pmatrix} x^2 \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ xy \end{pmatrix} <_{M(T)} \begin{pmatrix} xy \\ 0 \end{pmatrix} <_{M(T)} \cdots$$

while according to the index first ordering based on $<_T$ we get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ x \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ y \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ x^2 \end{pmatrix} <_{M(T)} \begin{pmatrix} 0 \\ xy \end{pmatrix} <_{M(T)} \cdots$$

$$<_{M(T)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} x \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} y \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} x^2 \\ 0 \end{pmatrix} <_{M(T)} \begin{pmatrix} xy \\ 0 \end{pmatrix} <_{M(T)} \cdots$$

For an chosen admissible ordering $<_{M(T)}$, we can uniquely represent any nonzero r-tuple of polynomial $\vec{f}$ as

$$\vec{f} = \sum_{i=1}^{\sigma} \mathrm{cf}(\vec{f}, \vec{p}_i) \cdot \vec{p}_i, \qquad \mathrm{cf}(\vec{f}, \vec{p}_i) \in K \setminus \{0\}, \quad \vec{p}_i \in P,$$

$$\vec{p}_1 <_{M(T)} \vec{p}_2 <_{M(T)} \cdots <_{M(T)} \vec{p}_\sigma. \tag{2.37}$$

where $\mathrm{cf}(\vec{f}, \vec{p}_i)$ is the *coefficient of* $\vec{p}_i$ in $\vec{f}$.

Further, the following notations with respect to the chosen ordering are defined.

$\mathrm{lppt}(\vec{f})$ the *leading power product tuple of* $\vec{f}$, i.e., $\vec{p}_\sigma$;

$\mathrm{lpp}(\vec{f})$ the *leading power product of* $\vec{f}$, i.e., the nonzero component of $\vec{p}_\sigma$;

$\mathrm{lcf}(\vec{f})$ the *leading coefficient of* $\vec{f}$, i.e., $\mathrm{cf}(\vec{f}, \vec{p}_\sigma)$;

$\mathrm{lt}(\vec{f})$ the *leading term of* $\vec{f}$, i.e., $\mathrm{lcf}(\vec{f}) \cdot \mathrm{lpp}(\vec{f})$;

$\mathrm{hp}(\vec{f})$ the *head position of* $\vec{f}$, i.e., if the nonzero component of $\vec{p}_\sigma$ occurs at the kth position, then $\mathrm{hp}(\vec{f}) = k$.

Similarly as for the polynomial case, the notions of reduction, normal form and Gröbner basis can be defined for elements of a module over $K[x_1, \ldots, x_n]$.

Definition 2.6 [Reduction]

Let $\vec{f}$, $\vec{g}$, $\vec{h} \in K^r[x_1, \ldots, x_n]$, $\vec{f} \neq \vec{0} \triangleq (0, \ldots, 0)$. *Then the reduction relation* $\vec{g} \rightarrow_{\vec{f}}$ *is defined as*

$$\vec{g} \rightarrow_{\vec{f}} \vec{h} \Leftrightarrow (\exists v,\ v \text{ power product})[\mathrm{cf}(\vec{g}, v \cdot \mathrm{lppt}(\vec{f})) \neq 0$$

$$\text{and } \vec{h} = \vec{g} - \frac{\mathrm{cf}(\vec{g}, v \cdot \mathrm{lppt}(\vec{f}))}{\mathrm{lcf}(\vec{f})} \cdot v \cdot \vec{f}]. \tag{2.38}$$

Let $F \subseteq K^r[x_1, \ldots, x_n]$. *Then* $\vec{h}$ *is a reduction of* $\vec{g}$ *modulo* F *and denoted by* $\vec{g} \rightarrow_F \vec{h}$ *if and only if there exists* $\vec{f} \in F$ *such that* $\vec{g} \rightarrow_{\vec{f}} \vec{h}$.

Definition 2.7 [Normal form]

Let $\vec{h} \in K^r[x_1, \ldots, x_n]$ *and* F *be a finite subset of* $K^r[x_1, \ldots, x_n]$. $\vec{h}$ *is normal form (or reduced form) modulo* F *if and only if there is no* $\vec{h}' \in K^r[x_1, \ldots, x_n]$ *such that* $\vec{h} \rightarrow_F \vec{h}'$.

Then $\vec{h}$ is a normal form of $\vec{f}$ modulo F, denoted by $\mathrm{NF}(F, \vec{f})$, *if and only if there is a sequence of reductions such that*

$$\vec{f} = \vec{k}_0 \rightarrow_F \vec{k}_1 \rightarrow_F \vec{k}_2 \rightarrow_F \cdots \rightarrow_F \vec{k}_q = \vec{h} \tag{2.39}$$

and $\vec{h}$ is in normal form modulo F. This reduction sequence will be denoted by $\vec{f} \xrightarrow{*}_F \vec{h}$.

A suitable generalization of the notion of S-polynomial is of great importance for the construction of Gröbner basis of a module, which is defined as follows.

Definition 2.8 [S-polynomial]

Let $\vec{f}$, $\vec{g} \in K^r[x_1, \dots, x_n]$. The S-polynomial of $\vec{f}$ and $\vec{g}$, denoted by $\mathrm{Sp}(\vec{f}, \vec{g})$, *is defined by*

$$\mathrm{Sp}(\vec{f}, \vec{g}) = \begin{cases} u \cdot \vec{f} - \dfrac{\mathrm{lcf}(\vec{f})}{\mathrm{lcf}(\vec{g})} \cdot v \cdot \vec{g} & \text{if } \mathrm{hp}(\vec{f}) = \mathrm{hp}(\vec{g}) \\ 0 & \text{otherwise} \end{cases} \tag{2.40}$$

where u, v are such that $\mathrm{lcm}(\mathrm{lpp}(\vec{f}), \mathrm{lpp}(\vec{g})) = u \cdot \mathrm{lpp}(\vec{f}) = v \cdot \mathrm{lpp}(\vec{g})$, *with* lcm *the least common multiple.*

Definition 2.9 [Gröbner basis]

Let M be a submodule of $K^r[x_1, \dots, x_n]$. A finite subset $G = \{\vec{g}_1, \dots, \vec{g}_t\}$ of $K^r[x_1, \dots, x_n]$ is a Gröbner basis of M if and only if the following conditions are satisfied:

i) *Module*$(G) = M$,

ii) *for any $\vec{f} \in M$, $\vec{f} \xrightarrow{*}_G \vec{0}$.*

Some important properties of Gröbner bases for modules in $K^r[x_1, \dots, x_n]$ are summarized by the following theorem:

Theorem 2.7 *Let M be a submodule of $K^r[x_1, \dots, x_n]$, $G = \{\vec{g}_1, \dots, \vec{g}_t\}$ a finite subset of M. The following statements are equivalent:*

GP1') *G is a Gröbner basis for M;*

GP2') *For all $\vec{f}$, $\vec{g} \in M$: $\vec{f} - \vec{g} \in M$ if and only if* $\mathrm{NF}(G, \vec{f}) = \mathrm{NF}(G, \vec{g})$;

GP3') *For all $\vec{g}_1$, $\vec{g}_2 \in G$:* $\mathrm{NF}(G, \mathrm{Sp}(\vec{g}_1, \vec{g}_2)) = 0$.

Similar to the polynomial case, GP2' corresponds to the property that if G is a Gröbner basis of M then for all $\vec{f} \in M$, $\mathrm{NF}(G, \vec{f}) = 0$. This fact implies that the approach of Gröbner basis for polynomial module can be directly applied to solve polynomial matrix equations [43, 105]. By GP3', a constructive algorithm can be established to calculate the Gröbner basis for a given module M [43, 73, 105].

Chapter 3

Description and Stability of Multidimensional Systems

With a view to making the present thesis self-contained, this chapter briefly summarizes some basic concepts and known results concerning the description and stability of nD discrete systems.

3.1 Multidimensional Signals and Systems

3.1.1 Multidimensional Signals

A *signal* can be defined as a function that conveys information, in general about the state or behavior of a physical system. Mathematically, an nD signal is represented as a function of n independent variables.

The independent variables and the amplitude of an nD signal may be either continuous or discrete. For these distinctions, nD signals are classified as continuous, discrete and digital signals. *Continuous signals* are signals whose independent variables are continuous and thus are represented by continuous variable functions. *Discrete signals* are those which possess discrete variables but continuous amplitudes and thus are characterized by sequences. *Digital signals* are those for which both independent variables and amplitudes are discrete. Additionally, signals whose independent variables and amplitudes are both continuous are sometimes referred to as *analog signals.*

Since the purpose of this paper is to consider various problems in nD discrete system theory, we are concerned in particular with processing nD discrete signals that are represented by sequences. A sequence x in n discrete variables (integer variables) $i_1, \ldots, i_n$ is

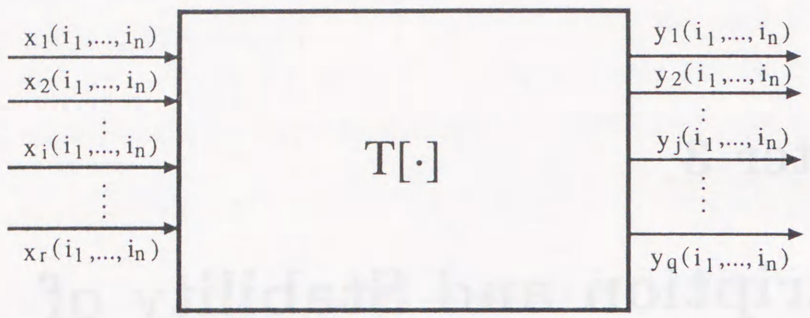


Figure 3.1: Representation of an MIMO nD System

formally expressed as

$$x = x(i_1, \ldots, i_n), \qquad -\infty < i_j < \infty, \quad j = 1, \ldots, n. \tag{3.1}$$

As in the 1D case, some typical nD discrete signals, such as unit-sample sequence, play important roles in nD system theory. The *unit-sample sequence* $\delta(z_1, \ldots, z_n)$, also often referred to as *unit impulse*, is defined as the sequence that is zero except at the origin, i.e.,

$$\delta(i_1, \ldots, i_n) = \begin{cases} 1, & i_1 = i_2 = \cdots = i_n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

3.1.2 Linear, Shift-Invariant nD Discrete Systems

A single-input single-output (SISO) nD *discrete system* is mathematically defined as a unique transformation or operator that maps an input nD discrete signal $x(i_1, \ldots, i_n)$ into an output nD discrete signal $y(i_1, \ldots, i_n)$. More generally, a multi-input multi-output (MIMO) nD discrete system can be viewed as a unique transformation or operator mapping several input nD discrete signals into several output nD discrete signals. Figure 3.1 illustrates a system that maps r inputs $x_1(i_1, \ldots, i_n), \ldots, x_r(i_1, \ldots, i_n)$ into q outputs $y_1(i_1, \ldots, i_n), \ldots, y_q(i_1, \ldots, i_n)$. The operator embodied in this system is represented by $T[\cdot]$, so by defining

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} \tag{3.3}$$

we may write

$$y(i_1, \ldots, i_n) = T[x(i_1, \ldots, i_n)]. \tag{3.4}$$

In general, the transformation $T[\cdot]$ may be rather complex. Due to different constraints imposed on $T[\cdot]$, classes of nD discrete systems can be specified. Among them, an important class is the *linear shift-invariant systems* defined as follows.

Definition 3.1 *An nD discrete system characterized by $T[\cdot]$ is said to be linear if and only if for any inputs $\boldsymbol{x}_1(i_1, \ldots, i_n)$, $\boldsymbol{x}_2(i_1, \ldots, i_n)$ and arbitrary constants c_1 and c_2,*

$$T[c_1\boldsymbol{x}_1(i_1, \ldots, i_n) + c_2\boldsymbol{x}_2(i_1, \ldots, i_n)] = c_1T[\boldsymbol{x}_1(i_1, \ldots, i_n)] + c_2T[\boldsymbol{x}_2(i_1, \ldots, i_n)]. \tag{3.5}$$

If $\boldsymbol{y}(i_1, \ldots, i_n)$ is the response to $\boldsymbol{x}(i_1, \ldots, i_n)$, i.e., $\boldsymbol{y}(i_1, \ldots, i_n) = T[\boldsymbol{x}(i_1, \ldots, i_n)]$, the system is said to be shift-invariant if and only if for all $\boldsymbol{x}(i_1, \ldots, i_n)$ and arbitrary integers $k_1, \ldots, k_n$,

$$\boldsymbol{y}(i_1 - k_1, \ldots, i_n - k_n) = T[\boldsymbol{x}(i_1 - k_1, \ldots, i_n - k_n)]. \tag{3.6}$$

The system that satisfies both the above properties is then linear shift-invariant (LSI).

It is possible to represent any nD vector sequence $\boldsymbol{x}(i_1, \ldots, i_n)$ as a sum of vectors whose nonzero elements are weighted and shifted nD unit impulses:

$$\begin{aligned} \boldsymbol{x}(i_1, \ldots, i_n) &= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \boldsymbol{x}(k_1, \ldots, k_n)\delta(i_1 - k_1, \ldots, i_n - k_n) \\ &= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \{x_1(k_1, \ldots, k_n)e_1\delta(i_1 - k_1, \ldots, i_n - k_n) + \\ &\qquad \cdots + x_r(k_1, \ldots, k_n)e_r\delta(i_1 - k_1, \ldots, i_n - k_n)\} \end{aligned} \tag{3.7}$$

where $\boldsymbol{e}_j \in \mathbf{R}^r$, $j = 1, \ldots, r$, denote the vectors having 1 as the element at the jth position and zeros at the other positions; $\delta(i_1 - k_1, \ldots, i_n - k_n)$ represents a shifted nD unit impulse and its nonzero sample is at $(k_1, \ldots, k_n)$; and the values $x_j(k_1, \ldots, k_n)$, $j = 1, \ldots, r$, can be interpreted as scalar multipliers for the corresponding unit impulses.

If we use this vector sequence as the input to nD discrete system $T[\cdot]$, then the output vector sequence determined by Equation (3.4) can be represented as:

$$\begin{aligned} \boldsymbol{y}(i_1, \ldots, i_n) &= T\Big[\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \{x_1(k_1, \ldots, k_n)e_1\delta(i_1 - k_1, \ldots, i_n - k_n) + \\ &\qquad \cdots + x_r(k_1, \ldots, k_n)e_r\delta(i_1 - k_1, \ldots, i_n - k_n)\}\Big] \\ &= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \{x_1(k_1, \ldots, k_n)T[e_1\delta(i_1 - k_1, \ldots, i_n - k_n)] + \end{aligned}$$

$$\cdots + x_r(k_1,\ldots,k_n)T[\boldsymbol{e}_r\delta(i_1-k_1,\ldots,i_n-k_n)]\}$$

$$= \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\{x_1(k_1,\ldots,k_n)\boldsymbol{h}^{(1)}_{k_1\ldots k_n}(i_1,\ldots,i_n) +$$

$$+\cdots+x_r(k_1,\ldots,k_n)\boldsymbol{h}^{(r)}_{k_1\ldots k_n}(i_1,\ldots,i_n)\}$$

$$= \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} H_{k_1\ldots k_n}(i_1,\ldots,i_n)\boldsymbol{x}(k_1,\ldots,k_n) \tag{3.8}$$

where

$$H_{k_1\ldots k_n}(i_1,\ldots,i_n) = [\boldsymbol{h}^{(1)}_{k_1\ldots k_n}(i_1,\ldots,i_n)\ \cdots\ \boldsymbol{h}^{(r)}_{k_1\ldots k_n}(i_1,\ldots,i_n)] \tag{3.9}$$

is a $q\times r$ matrix sequence.

For the special case where $k_1=\cdots=k_n=0$, we have

$$\boldsymbol{h}^{(j)}_{0\ldots0}(i_1,\ldots,i_n) = T[\boldsymbol{e}_j\delta(i_1,\ldots,i_n)], \qquad j=1,\ldots,r. \tag{3.10}$$

Applying the property of shift invariance shown by Equation (3.6), we conclude

$$\boldsymbol{h}^{(j)}_{k_1\ldots k_n}(i_1,\ldots,i_n) = \boldsymbol{h}^{(j)}_{0\ldots0}(i_1-k_1,\ldots,i_n-k_n), \qquad j=1,\ldots,r \tag{3.11}$$

Therefore, we can also get

$$H_{k_1\ldots k_n}(i_1,\ldots,i_n) = H_{0\ldots0}(i_1-k_1,\ldots,i_n-k_n) \tag{3.12}$$

Defining

$$H(i_1,\ldots,i_n) = H_{0\ldots0}(i_1,\ldots,i_n), \tag{3.13}$$

we can then write the output vector sequence as

$$\boldsymbol{y}(i_1,\ldots,i_n) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} H(i_1-k_1,\ldots,i_n-k_n)\boldsymbol{x}(k_1,\ldots,k_n) \tag{3.14}$$

Suppose that $r=q=1$, then Equation (3.14) corresponds to the input/output relation of an SISO nD system. Applying an nD unit impulse as input to the system, the output sequence will be

$$\{y(i_1,\ldots,i_n)\mid y(i_1,\ldots,i_n) = H(i_1,\ldots,i_n),\ (i_1,\ldots,i_n)\in Z^n_+\}. \tag{3.15}$$

For this reason the sequence $\{H(i_1,\ldots,i_n)\mid(i_1,\ldots,i_n)\in Z^n_+\}$ is called the *impulse response* of the SISO system.

Similarly to the scalar case the matrix sequence $\{H(i_1,\ldots,i_n)\mid(i_1,\ldots,i_n)\in Z^n_+\}$ is also called the impulse response for the general MIMO case of system (3.14). In fact, the reasonableness for this definition can be also shown as follows. It is easy to see that the ijth element $h_{ij}(i_1,\ldots,i_n)$ of $H(i_1,\ldots,i_n)$ is the impulse response at the ith output port when the jth input signal is an nD unit impulse $\delta(i_1,\ldots,i_n)$ and all the other input signals are zero. In general, of course, there will be arbitrary signals at every input port of the system, whence the ith output signal must be

$$y_i(i_1,\ldots,i_n) = \textstyle\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\{\sum_{j=1}^{r} h_{ij}(i_1-k_1,\ldots,i_n-k_n)x_j(k_1,\ldots,k_n)\},$$
$$i+1,\ldots,q \tag{3.16}$$

in view of the principles of superposition and shift invariance. This relation is obviously identical to Equation (3.14). The relation given by Equation (3.14) is referred to as nD *convolution sum*, and it implies that an nD LSI system is completely characterized by its impulse response.

An important class of nD LSI systems is those whose input/output relations satisfy a set of, say s, linear difference equations which can be represented in the matrix form as

$$\mathop{\sum_{i_1}\cdots\sum_{i_n}}_{(i_1,\ldots,i_n)\in D_a} A(i_1,\ldots,i_n)\boldsymbol{y}(k_1-i_1,\ldots,k_n-i_n) =$$
$$\mathop{\sum_{i_1}\cdots\sum_{i_n}}_{(i_1,\ldots,i_n)\in D_b} B(i_1,\ldots,i_n)\boldsymbol{x}(k_1-i_1,\ldots,k_n-i_n) \tag{3.17}$$

where $A(i_1,\ldots,i_n)\in\mathbf{R}^{s\times q}$, $B(i_1,\ldots,i_n)\in\mathbf{R}^{s\times r}$, and D_a and D_b are suitable output and input masks, respectively, denoting the finite regions of support for coefficient arrays $\{A(i_1,\ldots,i_n)\}$ and $\{B(i_1,\ldots,i_n)\}$.

Difference equations serve not only as means for defining certain LSI systems, but also as computational algorithms for realizing those systems. For, if $s=q$ and $\det A(0,\ldots,0)\neq 0$, Equation (3.17) can be written as

$$\boldsymbol{y}(k_1,\ldots,k_n) = A^{-1}(0,\ldots,0)\{\mathop{\sum_{i_1}\cdots\sum_{i_n}}_{(i_1,\ldots,i_n)\in D_a} B(i_1,\ldots,i_n)\boldsymbol{x}(k_1-i_1,\ldots,k_n-i_n)$$

$$- \sum_{\substack{i_1 \\ (i_1,\ldots,i_n)\in D_b \\ (i_1,\ldots,i_n)\neq(0,\ldots,0)}} \cdots \sum_{i_n} A(i_1,\ldots,i_n)y(k_1-i_1,\ldots,k_n-i_n)\}. \quad (3.18)$$

In this form, the output sample $y(k_1,\ldots,k_n)$ can be computed provided that the required input samples are available and that the output samples which occurs on the right-hand side of Equation (3.18) have either been previously computed or have been specified as initial conditions. The systems for which the output samples can be computed in this manner are said to be *recursively computable.*

3.1.3 Causality of nD Systems

In 1D systems, the independent variable is usually *time*. It is then natural and useful to characterize the class of 1D *causal systems* for which the outputs could not precede the inputs, in view of both the physical signification of the constraint and the possibility of real time implementation of the systems. In fact, a 1D LSI system is causal if and only if its impulse response $H(i)$ is zero for $i < 0$.

Since for most nD systems the independent variables do not correspond to time, causality is not a natural constraint or an intrinsic property of such systems. Nevertheless, it is in fact useful to generalize the concept of causality to nD systems for the motivation of recursive computable implementation. Such a generalization can be done by requiring that an impulse response to be zero outside some region of support. Most commonly, an nD LSI system is defined to be *causal* if its impulse response is zero outside the closed first quadrant of $\mathbf{R}^n$, i.e., $H(i_1,\ldots,i_n) = 0$ if any $i_j < 0$, $j \in \{1,2,\ldots,n\}$ [29, 32, 49]. (An alternative definition of causality in terms of rational functions will be given in Section 3.3.)

In nD system theory, another class of systems is characterized as *weakly causal systems.* Since we will only consider the (first-quadrant) causal systems in this thesis, we just refer the interested readers to [32, 49].

3.2 nD $\mathcal{Z}$-Transformation

As in the 1D case, z-transform is a useful mathematical tool in studying nD LSI systems.

The nD z-transform is generally defined in the literature (see, e.g., [20, 51]) as

$$\mathcal{Z}[f(i_1,\ldots,i_n)] \triangleq F(z_1,\ldots,z_n) = \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} f(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n} \quad (3.19)$$

where $z_1,\ldots,z_n$ can be regarded as delay operators. An alterative definition can also be given by replacing $z_1^{i_1}\cdots z_n^{i_n}$ with $z_1^{-i_1}\cdots z_n^{-i_n}$ in Equation (3.19) (see, e.g., [29, 54]). There is no, however, substantial conceptual difference between the two possibilities [20].

For causal LSI systems, the nD z-transform defined by Equation (3.19) turns out to be

$$F(z_1,\ldots,z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n}. \quad (3.20)$$

Detailed discussions on the properties of nD z-transform can be found in, e.g., [20, 29, 54].

3.3 Transfer Function Representation of nD Systems

As mentioned in Section 3.1, the input/output relation of an nD LSI system can be represented by means of a convolution sum as in Equation (3.14). Applying nD z-transform defined by Equation (3.20) to a causal LSI system described by the convolution sum, we get

$$Y(z_1,\ldots,z_n) = H(z_1,\ldots,z_n)X(z_1,\ldots,z_n) \quad (3.21)$$

where

$$Y(z_1,\ldots,z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} y(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n}, \quad (3.22)$$

$$X(z_1,\ldots,z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} x(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n}, \quad (3.23)$$

$$H(z_1,\ldots,z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} H(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n}. \quad (3.24)$$

$H(z_1,\ldots,z_n)$ is referred to as the *transfer (function) matrix* of the system, and it can be regarded either as a formal power series or as a function of $z_1,\ldots,z_n$ when the power series converges.

For the systems specified by a set of difference equations as in Equation (3.17), we have the result of z-transform as

$$A(z_1, \ldots, z_n)Y(z_1, \ldots, z_n) = B(z_1, \ldots, z_n)X(z_1, \ldots, z_n) \tag{3.25}$$

where

$$A(z_1, \ldots, z_n) = \sum_{\substack{i_1 \\ (i_1,\ldots,i_n) \in D_a}} \cdots \sum_{i_n} A(i_1, \ldots, i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{3.26}$$

$$B(z_1, \ldots, z_n) = \sum_{\substack{i_1 \\ (i_1,\ldots,i_n) \in D_b}} \cdots \sum_{i_n} B(i_1, \ldots, i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{3.27}$$

Since the coefficient arrays $A(i_1, \ldots, i_n)$ and $B(i_1, \ldots, i_n)$ have finite area of support so that their z-transforms are simply nD polynomial matrices.

When $A(z_1, \ldots, z_n)$ is square ($q \times q$) and $\det A(z_1, \ldots, z_n) \neq 0$, the input/output relation of Equation (3.25) is said to be *regular*. In what follows, we will only consider regular nD discrete systems. Under this assumption, Equation (3.25) can written as

$$\begin{aligned} Y(z_1, \ldots, z_n) &= A^{-1}(z_1, \ldots, z_n)B(z_1, \ldots, z_n)X(z_1, \ldots, z_n) \\ &= H(z_1, \ldots, z_n)X(z_1, \ldots, z_n) \end{aligned} \tag{3.28}$$

where the transfer matrix

$$H(z_1, \ldots, z_n) = A^{-1}(z_1, \ldots, z_n)B(z_1, \ldots, z_n) \tag{3.29}$$

is a $q \times r$ matrix with entries of rational functions in $z_1, \ldots, z_n$. A rational function description of $H(z_1, \ldots, z_n)$ is desired from the standpoint of recursive implementation of the input/output relation.

For the 2D case, a rational transfer matrix $H(z_1, z_2)$ is said to be *causal* if and only if its entries $h_{ij}(z_1, z_2)$ satisfy that $h_{ij}(z_1, z_2) = b_{ij}(z_1, z_2)/a_{ij}(z_1, z_2)$ with $a_{ij}(z_1, z_2), b_{ij}(z_1, z_2) \in \mathbf{R}[z_1, z_2]$ and $a_{ij}(0, 0) \neq 0$; and is said to be *strictly causal* if in addition $b_{ij}(0, 0) = 0$ [30, 49]. Further, it is known that $H(z_1, z_2)$ is causal if and only if the associated input/output system (3.14) is causal in the sense defined in Section 3.1 [30, 49].

Analogously, in this thesis, an nD rational transfer matrix $H(z_1, \ldots, z_n)$ having entries $h_{ij}(z_1, \ldots, z_2) = b_{ij}(z_1, \ldots, z_n)/a_{ij}(z_1, \ldots, z_n)$ where $a_{ij}(z_1, \ldots, z_n)$, $b_{ij}(z_1, \ldots, z_n) \in$

$\mathbf{R}[z_1, \ldots, z_n]$ is also referred to as *causal* if and only if $a_{ij}(0, \ldots, 0) \neq 0$.

3.4 State-Space Representation of nD Systems

State space representation of nD systems, particularly of 2D systems, have been intensively investigated and there are quite a few of contributions in the literature (see [4, 38, 40, 71, 74, 81] and the references therein). Attasi [4], Fornasini-Marchesini [38] and Roesser [81] proposed different state-space models for 2D systems, respectively. It has been proved that by using the models introduced by Roesser and Fornasini-Marchesini, the whole class of causal rational transfer functions can be realized, whereas the model proposed by Attasi realizes only the subclass of *separable*, or *recognizable*, transfer functions [37, 39, 40]. It has also been shown that the Attasi model can be embedded in the Roesser model and Fornasini-Marchesini model, and the latter two can be embedded in each other [40, 74]. However, it was noted that the embedding of the Roesser model into Fornasini-Marchesini model preserves the dimension of the local state space, while the reverse embedding requires in general increasing the dimension of the local state space [40]. The 2D Roesser model has been extended to 3D systems by Tzafestas-Pimenides [102] and Theordoru-Tzafestas [99], and more generally to nD systems by Kurek [65].

A major difference from the 1D case is that the state-space of nD systems is of infinite dimension, and this results in the necessity to introduce the new concepts (finitely dimensional) *local* state space and (infinitely dimensional) *global* state space [38, 64]. That is, while very compact nD state-space models are based on local state space which preserves finite past information needed for the recursions, it is usually also necessary to take into account the global state of a system which preserves all (infinite) past information.

These properties, therefore, make it rather subtle and difficult to extend to the nD case the well-known 1D notions such as controllability, observability and minimality. Indeed, up to now, various notions have been suggested from different standpoints in the literature (see, e.g. [31, 38, 59, 64, 81]).

As stated earlier, the attention of this thesis will be concentrated to algebra approaches. Hence, we do not repeat all these concepts and results, but just recall some fundamental results on nD Roesser model, which will be used later in the thesis.

3.4.1 nD Roesser State-Space Model

Roesser state-space model for an MIMO nD system [54, 65] is given in the form

$$\begin{cases} \boldsymbol{x}'(i_1,\ldots,i_n) = A\boldsymbol{x}(i_1,\ldots,i_n) + B\boldsymbol{u}(i_1,\ldots,i_n) \\ \boldsymbol{y}(i_1,\ldots,i_n) = C\boldsymbol{x}(i_1,\ldots,i_n) + D\boldsymbol{u}(i_1,\ldots,i_n) \end{cases} \tag{3.30}$$

where $\boldsymbol{u}(i_1,\ldots,i_n) \in \mathbf{R}^m$ and $\boldsymbol{y}(i_1,\ldots,i_n) \in \mathbf{R}^l$ are the input and output vectors, respectively; $\boldsymbol{x}(i_1,\ldots,i_n) \in \mathbf{R}^{\tilde{n}}$ is the local state vector in the form

$$\boldsymbol{x}(i_1,\ldots,i_n) = \begin{bmatrix} \boldsymbol{x}_1(i_1,\ldots,i_n) \\ \boldsymbol{x}_2(i_1,\ldots,i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1,\ldots,i_n) \end{bmatrix}, \quad \boldsymbol{x}'(i_1,i_2,\ldots,i_n) = \begin{bmatrix} \boldsymbol{x}_1(i_1+1,i_2,\cdots,i_n) \\ \boldsymbol{x}_2(i_1,i_2+1,\cdots,i_n) \\ \vdots \\ \boldsymbol{x}_n(i_1,i_2,\cdots,i_n+1) \end{bmatrix}$$

with $\boldsymbol{x}_i(i_1,\ldots,i_n) \in \mathbf{R}^{n_i}$ $(i = 1,\ldots,n,\ \tilde{n} = \sum_{i=1}^{n} n_i)$ being the ith (sub-)state vector of $\boldsymbol{x}(i_1,\ldots,i_n)$; and

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

$$C = [C_1\, C_2\, \cdots\, C_n]$$

with A_{ij}, B_i, C_i and D being constant matrices of suitable dimensions, particularly, $A_{ii} \in \mathbf{R}^{n_i \times n_i}$.

It is assumed that the boundary conditions are given by $\{\boldsymbol{x}_j(i_1,\ldots,0_j,\ldots,i_n) \mid i_k \in Z_+,\ k,\ j = 1,\ldots,n\}$.

3.4.2 Transition Matrices and General Responses

For integer n-tuple $(i_1,\ldots,i_n)$ the following partial ordering is defined:

$$(i_1,\ldots,i_j,\ldots,i_n) \le (k_1,\ldots,k_j,\ldots,k_n) \iff i_j \le k_j,\ j = 1,\ldots,n$$

$$(i_1,\ldots,i_j,\ldots,i_n) = (k_1,\ldots,k_j,\ldots,k_n) \iff i_j = k_j,\ j = 1,\ldots,n$$

$$(i_1,\ldots,i_n) < (k_1,\ldots,k_n) \iff (i_1,\ldots,i_n) \le (k_1,\ldots,k_n)$$
$$\text{and } (i_1,\ldots,i_n) \ne (k_1,\ldots,k_n)$$

The notation $(i_1,\ldots,0_j,\ldots,i_n)$ denotes the n-tuple with zero on the j-th position. Similarly, the notation $(0,\ldots,1_j,\ldots,0)$ denotes the n-tuple with one on the j-th position and zero elsewhere.

Then, the *(state) transition matrix* $A^{i_1,\ldots,i_j,\ldots,i_n}$ for nD system (3.30) is defined as follows [54, 59, 65]:

$$A^{0,\ldots,0_j,\ldots,0} = I \tag{3.31}$$

$$A^{0,\ldots,1_j,\ldots,0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{j1} & A_{j2} & \cdots & A_{jn} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.32}$$

$$A^{i_1,\ldots,i_j,\ldots,i_n} = \sum_{j=1}^{n} A^{0,\ldots,1_j,\ldots,0} A^{i_1,\ldots,i_j-1,\ldots,i_n} \tag{3.33}$$

$$A^{i_1,\ldots,i_j,\ldots,i_n} = 0 \quad \forall i_j < 0,\ j = 1,\ldots,n \tag{3.34}$$

By using the state-transition matrix $A^{i_1,\ldots,i_j,\ldots,i_n}$, the response formula of the nD system, i.e., the solution to the equation (3.30) can be given in terms of the inputs and the boundary conditions [54, 59, 65]:

$$\begin{aligned} \boldsymbol{x}(i_1,\ldots,i_n) = &\sum_{j=1}^{n} \sum_{\substack{(i_1,\ldots,i_j,\ldots,i_n) \\ \ge (k_1,\ldots,k_j,\ldots,k_n) \\ \ge (0,\ldots,0_j,\ldots,0)}} A^{i_1-k_1,\ldots,i_j,\ldots,i_n-k_n} \left[0 \cdots \boldsymbol{x}_j^T(k_1,\ldots,0_j,\ldots,k_n) \cdots 0\right]^T \\ &+ \sum_{\substack{(i_1,\ldots,i_j,\ldots,i_n) \\ > (k_1,\ldots,k_j,\ldots,k_n) \\ \ge (0,\ldots,0_j,\ldots,0)}} \left(\sum_{j=1}^{n} A^{i_1-k_1,\ldots,i_j-k_j-1,\ldots,i_n-k_n} B^{0,\ldots,1_j,\ldots,0} \right) \boldsymbol{u}(k_1,\ldots,k_j,\ldots,k_n) \end{aligned} \tag{3.35}$$

where $B^{0,\ldots,1_j,\ldots,0}$ is defined as

$$B^{0,\ldots,1_j,\ldots,0} = [0 \cdots B_j^T \cdots 0]^T, \qquad j = 1,2,\ldots,n. \tag{3.36}$$

3.4.3 Transfer Function Matrix of the Roesser Model

Applying nD $\mathcal{Z}$-transform to (3.30) gives

$$\boldsymbol{x}(z_1,\ldots,z_n) = (I - ZA)^{-1} ZB\boldsymbol{u}(z_1,\ldots,z_n) + (I - ZA)^{-1} \tilde{\mathcal{X}}_0 \tag{3.37}$$

$$y(z_1,\ldots,z_n) = Cx(z_1,\cdots,z_n) + Du(z_1,\ldots,z_n) \tag{3.38}$$

where

$$Z = \begin{bmatrix} I_{m_1}z_1 & 0 & \cdots & 0 \\ 0 & I_{m_2}z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_n}z_n \end{bmatrix} \tag{3.39}$$

and

$$\tilde{\mathcal{X}}_0 = \begin{bmatrix} \sum_{i_2,\ldots,i_k=0}^{\infty} x_1(0,i_2,\ldots,i_n) z_2^{i_2}\ldots z_n^{i_n} \\ \sum_{i_1,i_3,\ldots,i_n=0}^{\infty} x_2(i_1,0,i_3,\ldots,i_n) z_1^{i_1} z_3^{i_3}\ldots z_n^{i_n} \\ \vdots \\ \sum_{i_1,\ldots,i_{n-1}=0}^{\infty} x_n(i_1,\ldots,i_{n-1},0) z_1^{i_1}\ldots z_{n-1}^{i_{n-1}} \end{bmatrix} \tag{3.40}$$

When the boundary conditions are zero that implies $\tilde{\mathcal{X}}_0 = 0$, we get from (3.37) and (3.38) the following input/output relation:

$$y(z_1,\ldots,z_n) = G(z_1,\ldots,z_n)u(z_1,\ldots,z_n) \tag{3.41}$$

where

$$G(z_1,\ldots,z_n) = C(I - ZA)^{-1}ZB + D \tag{3.42}$$

is the transfer matrix of the nD system (3.30).

The characteristic polynomial of the nD system (3.30) is defined as follows [5].

$$\begin{aligned} \rho(z_1,\ldots,z_n) &= \det(I - ZA) \\ &= \det \begin{bmatrix} I_{m_1}-z_1A_{11} & -z_1A_{12} & \cdots & -z_1A_{1n} \\ -z_2A_{21} & I_{m_2}-z_2A_{22} & \cdots & -z_2A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -z_nA_{n1} & -z_nA_{n2} & \cdots & I_{m_n}-z_nA_{nn} \end{bmatrix} \end{aligned} \tag{3.43}$$

3.5 Stability of nD Systems

Owing to its great importance in system theory, the stability problem has been investigated very extensively, and quite a few results have been obtained (see, e.g., [5, 6, 20, 35, 41, 42, 46, 50, 51, 96, 98]). Therefore, we do not go far over the detail here but just document some very fundamental results for the background of our research.

3.5.1 BIBO Stability of nD Systems

Among the various possible definitions of stability, the bounded-input bounded-output (BIBO) stability is the most commonly used one. Since the BIBO stability of MIMO nD systems is essentially related to the one of SISO systems, for brevity we first consider the case for SISO systems described by the following input/output relation:

$$y(i_1,\ldots,i_n) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} h(i_1-k_1,\ldots,i_n-k_n)u(k_1,\ldots,k_n) \tag{3.44}$$

where $u(i_1,\ldots,i_n)$, $y(i_1,\ldots,i_n)$ and $h(i_1,\ldots,i_n)$ are the input, the output and the impulse response of the system, respectively.

Definition 3.2 (see, e.g., [20, 51])

An nD system described by Equation (3.44) is said to be BIBO stable if and only if, for all input signals $u(i_1,\ldots,i_n)$ such that

$$|u(i_1,\ldots,i_n)| \le M < \infty \quad \forall (i_1,\ldots,i_n) \in Z^n \tag{3.45}$$

where M is a finite real number, there exists a finite real number L such that, for the output $y(i_1,\ldots,i_n)$ of the system,

$$|y(i_1,\ldots,i_n)| \le L < \infty \tag{3.46}$$

holds.

For a causal LSI nD system, it is well-known that the BIBO stability coincides with the condition

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} |h(i_1,\ldots,i_n)| < \infty. \tag{3.47}$$

namely, the impulse response of the system is absolutely summable [51].

As in the 1D case, it is possible to formulate the relation between the BIBO stability and the singularities of the nD transfer function

$$g(z_1,\ldots,z_n) = \frac{n(z_1,\ldots,z_n)}{d(z_1,\ldots,z_n)}. \tag{3.48}$$

For nD ($n > 1$) systems, however, such relations are much more complicated than expected and unfortunately the formulation of such stability conditions does not directly provide an efficient stability test, as in the 1D case. This is mainly due to the possible existence of the nonessential singularities of the second kind and the fact that the singularities of a 1D polynomial are isolated or distinct points and those of an nD polynomial are usually nD surfaces or manifolds.

Before proceeding, let us first recall that the nonessential singularities of the first kind of a transfer function $g(z_1, \ldots, z_n)$ are also often referred to as the *poles* of $g(z_1, \ldots, z_n)$ (see, e.g., [20]).

Theorem 3.1 [46]

Consider a 2D LSI system described by the transfer function $g(z_1, z_2) = n(z_1, z_2)/d(z_1, z_2)$ where $n(z_1, z_2)$ and $d(z_1, z_2)$ are factor coprime, then,

(i) *the 2D system is BIBO stable if*

$$d(z_1, z_2) \neq 0, \qquad \forall (z_1, z_2) \in \bar{U}^2 \tag{3.49}$$

(ii) *if the 2D system $g(z_1, z_2)$ is BIBO stable then $g(z_1, z_2)$ has no poles in $\bar{U}^2$ and no nonessential singularities of the second kind in $\bar{U}^2$ except possibly on T^2.*

Theorem 3.2 [98]

Consider an nD LSI system described by the transfer function $g(z_1, \ldots, z_n) = n(z_1, \ldots, z_n)/d(z_1, \ldots, z_n)$, where $n(z_1, \ldots, z_n)$ and $d(z_1, \ldots, z_n)$ are factor coprime, then,

(i) *the nD system is BIBO stable if*

$$d(z_1, \ldots, z_n) \neq 0, \qquad \forall (z_1, \ldots, z_n) \in \bar{U}^n \tag{3.50}$$

(ii) *if the nD system $g(z_1, \ldots, z_n)$ is BIBO stable then $g(z_1, \ldots, z_n)$ has no poles in $\bar{U}^n$ and no nonessential singularities of the second kind in U^n.*

It is worth noting the following facts revealed by the above theorems. In the 1D case the absence of singularities in the unitdisc is known as a necessary and sufficient condition for the 1D BIBO stability. But the similar condition, i.e., the absence of singularities in the unit polydisc is only sufficient but no longer necessary for the nD ($n > 1$) cases. Furthermore, it should be noted that the situation is even quite different for the cases of $n = 2$ and $n \geq 3$. This is apparent from the fact that a 2D system may be BIBO stable even if it has nonessential singularities of the second kind on T^2 while an nD ($n \geq 3$) system may be BIBO stable even when it has nonessential singularities of the second kind on $\bar{U}^n - U^n$, i.e., not only on T^n but also on $(\bar{U}^n - U^n) - T^n$ [98].

Moreover, for MIMO nD systems, the following theorem can be established.

Theorem 3.3 [51]

An MIMO nD LSI system whose transfer function matrix is given by $G(z_1, \ldots, z_n) \in \mathbf{M}(\mathbf{R}(z_1, \ldots, z_n))$ is BIBO stable if and only if each entry of $G(z_1, \ldots, z_n)$ corresponds to a SISO system which is BIBO stable.

3.5.2 Structural Stability of nD Systems

In the study of nD LSI systems, it is convenient to introduce the notion of structural stability, which is slightly stronger than that of BIBO stability.

Definition 3.3 [49, 51]

A SISO nD system $g(z_1, \ldots, z_n) = n(z_1, \ldots, z_n)/d(z_1, \ldots, z_n)$, with $n(z_1, \ldots, z_n)$ and $d(z_1, \ldots, z_n)$ being factor coprime, is said to be structurally stable if and only if $d(z_1, \ldots, z_n)$ has no zeros in $\bar{U}^n$; while an MIMO nD systems $G(z_1, \ldots, z_n) \in \mathbf{M}(\mathbf{R}(z_1, \ldots, z_n))$ is said to be structurally stable if and only if each entry of $G(z_1, \ldots, z_n)$ corresponds to a SISO system which is structurally stable.

3.5.3 Internal Stability of 2D Systems

The internal stability of 2D systems was first investigated in [40, 41] based on Fornasini-Marchesini's state-space model, and later in [2] with respect to Roesser's model.

Let $n = 2$ in the system (3.30), $X = \mathbf{R}^{\bar{n}}$ be the local state space, and introduce the following notion:

$$\mathcal{X}_r = \{x(i_1, i_2) \mid x(i_1, i_2) \in X,\ i_1 + i_2 = r\}. \tag{3.51}$$

Then, denote by $\|x(i_1, i_2)\|$ the Euclidean norm of $x(i_1, i_2)$ in X and define

$$\|\mathcal{X}_r\| = \sup_{n \in Z_+} \|x(r - n, n)\|. \tag{3.52}$$

Definition 3.4 [2]

Let $\sum$ be a 2D system described by (3.30) with $n = 2$. Then $\sum$ is asymptotically stable if, assuming $u(i_1, i_2) = 0$ and $\|\mathcal{X}_0\|$ finite, $\|\mathcal{X}_i\| \to 0$ as $i \to +\infty$.

The following theorem shows a necessary and sufficient condition for 2D internal stability in term of the singularities of the system characteristic polynomial. For corresponding theorem with respect to Fornasini-Marchesini's model, we refer to [40, 41].

Theorem 3.4 [2]

Let $\sum$ be as defined above. Then $\sum$ is asymptotically stable if and only if the characteristic polynomial of $\sum$ is devoid of zeros in the unit bidisc, i.e.,

$$\det \begin{bmatrix} I - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I - z_2 A_{22} \end{bmatrix} \neq 0, \quad \forall (z_1, z_2) \in \bar{U}^2. \tag{3.53}$$

It is noted that if $\sum$ is a realization of a transfer function $g(z_1, z_2) = n(z_1, z_2)/d(z_1, z_2)$, then the condition $d(z_1, z_2) \neq 0$ in $\bar{U}^2$ is *both necessary and sufficient* for the internal stability of $g(z_1, z_2)$. A detailed discussion on the connections between BIBO stability and internal stability can be found in [10].

3.5.4 Stability Theorems

As shown above, to test stability of an nD system the main concern is to see whether or not a given denominator polynomial is devoid of zeros in the unit polydisc $\bar{U}^n$. In general, however, this is a formidable task. Some fundamental theorems will be documented here which can help to a certain extent in simplifying the nD stability tests.

For the 2D case, Huang [50] proposed the following theorem, correct proofs of which can be found in [25, 45].

Theorem 3.5 *Let $d(z_1, z_2) \in \mathbf{R}[z_1, z_2]$. Then, $d(z_1, z_2) \neq 0$, $\forall (z_1, z_2) \in \bar{U}^2$ if and only if*

(i) $d(z_1, z_2) \neq 0, \ |z_1| \leq 1, \ |z_2| = 1$

(ii) $d(a, z_2) \neq 0$ *for any a such that* $|a| \leq 1$

DeCarlo *et al.* [26] and Strintzis [97] independently simplified Huang's test and gave the criterion stated below.

Theorem 3.6 *Let $d(z_1, z_2) \in \mathbf{R}[z_1, z_2]$. Then, $d(z_1, z_2) \neq 0$, $\forall (z_1, z_2) \in \bar{U}^2$ if and only if*

(i) $d(z_1, z_2) \neq 0$, *for* $|z_1| = 1$ *and* $|z_2| = 1$

(ii) $d(a, z_2) \neq 0$ *for* $|z_2| \leq 1$ *and for any a such that* $|a| = 1$

(iii) $d(z_1, b) \neq 0$ *for* $|z_1| \leq 1$ *and for any b such that* $|b| = 1$

In particular, one can choose $a = b = 1$.

Still another criterion was developed by DeCarlo *et al.* [26] and it is given as follows.

Theorem 3.7 *Let $d(z_1, z_2) \in \mathbf{R}[z_1, z_2]$. Then, $d(z_1, z_2) \neq 0$, $\forall (z_1, z_2) \in \bar{U}^2$ if and only if*

(i) $d(z_1, z_2) \neq 0$, *for $z_1 = z_2 = z$ when* $|z| \leq 1$

(ii) $d(z_1, z_2) \neq 0$ *for* $|z_1| = |z_2| = 1$.

Anderson and Jury [3] generalized Huang's 2D stability test to the nD case as follows.

Theorem 3.8 *Let $d(z_1, \ldots, z_n) \in \mathbf{R}[z_1, \ldots, z_n]$. Then, $d(z_1, \ldots, z_n) \neq 0$, $\forall (z_1, \ldots, z_n) \in \bar{U}^n$ if and only if*

$$\begin{aligned} d(z_1, 0, \ldots, 0) &\neq 0, \quad |z_1| \leq 1 \\ d(z_1, z_2, 0, \ldots, 0) &\neq 0, \quad |z_1| = 1 \ \textit{and} \ |z_2| \leq 1 \\ &\vdots \\ d(z_1, z_2, \ldots, z_{n-1}, 0) &\neq 0, \quad |z_1| = \cdots = |z_{n-2}| = 1 \ \textit{and} \ |z_{n-1}| \leq 1 \\ d(z_1, z_2, \ldots, z_{n-1}, z_n) &\neq 0, \quad |z_1| = \cdots = |z_{n-1}| = 1 \ \textit{and} \ |z_n| \leq 1 \end{aligned}$$

DeCarlo *et al.* [26] and Strintzis [97] also developed their 2D stability test to the following nD version.

Theorem 3.9 *Let $d(z_1, \ldots, z_n) \in \mathbf{R}[z_1, \ldots, z_n]$. Then, $d(z_1, \ldots, z_n) \neq 0$, $\forall (z_1, \ldots, z_n) \in \bar{U}^n$ if and only if*

(i) $d(z_1, \ldots, z_n) \neq 0$, *for* $|z_1| = \cdots = |z_n| = 1$

(ii) $d(1, \ldots, 1, z_i, 1, \ldots, 1) \neq 0$ *for* $|z_i| \leq 1$ *for all* $i = 1, \ldots, n$

Further, DeCarlo *t al.* [26] and Murray [76] generalized the criterion of Theorem 3.7, which is simpler than the above.

Theorem 3.10 *Let $d(z_1, \ldots, z_n) \in \mathbf{R}[z_1, \ldots, z_n]$. Then, $d(z_1, \ldots, z_n) \neq 0$, $\forall (z_1, \ldots, z_n) \in \bar{U}^n$ if and only if*

(i) $d(z, \ldots, z) \neq 0$, *for* $|z| = 1$

(ii) $d(z_1, \ldots, z_n) \neq 0$ *for* $|z_1| = \cdots = |z_n| = 1$.

3.6 Some Examples of nD Control Systems

3.6.1 Partial-differential Systems [54, 70]

Consider the equation

$$\partial T(x,t)/\partial x = -\partial T(x,t)/\partial t - T(x,t) + U(t) \tag{3.54}$$

with initial and boundary conditions

$$T(x,0) = f_1(x), \qquad T(0,t) = f_2(t) \tag{3.55}$$

where $T(x,t)$ is unknown function (usually the temperature) at x(space) $\in [0, x_f]$ and t(time) $\in [0,\infty]$, $U(t)$ is a given force function and $f_1(x)$, $f_2(t)$ are given functions.

The equation (3.54) describes some thermal processes, for example, in chemical reactors, heat exchangers and pipe furnaces (see Figure 3.2). Let

$$T(i,j) = T(i\Delta x, j\Delta t), \qquad U(j) = U(j\Delta t)$$

$$\partial T(x,t)/\partial t \approx (T(i,j+1) - T(i,j))/\Delta t$$

$$\partial T(x,t)/\partial x \approx (T(i,j) - T(i-1,j))/\Delta x$$

Then we can write Equation (3.54) in the form

$$T(i,j+1) = a_1 T(i,j) + a_2 T(i-1,j) + bU(j) \tag{3.56}$$

where

$$a_1 = 1 - \Delta t/\Delta x - \Delta t, \qquad a_2 = \Delta t/\Delta x, \qquad b = \Delta t.$$

By defining

$$x_h(i,j) = T(i-1,j), \qquad x_v(i,j) = T(i,j) \tag{3.57}$$

we can represent Equation (3.56) by the Roesser's model

$$\begin{bmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} U(j) \tag{3.58a}$$

$$T(i,j) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} \tag{3.58b}$$

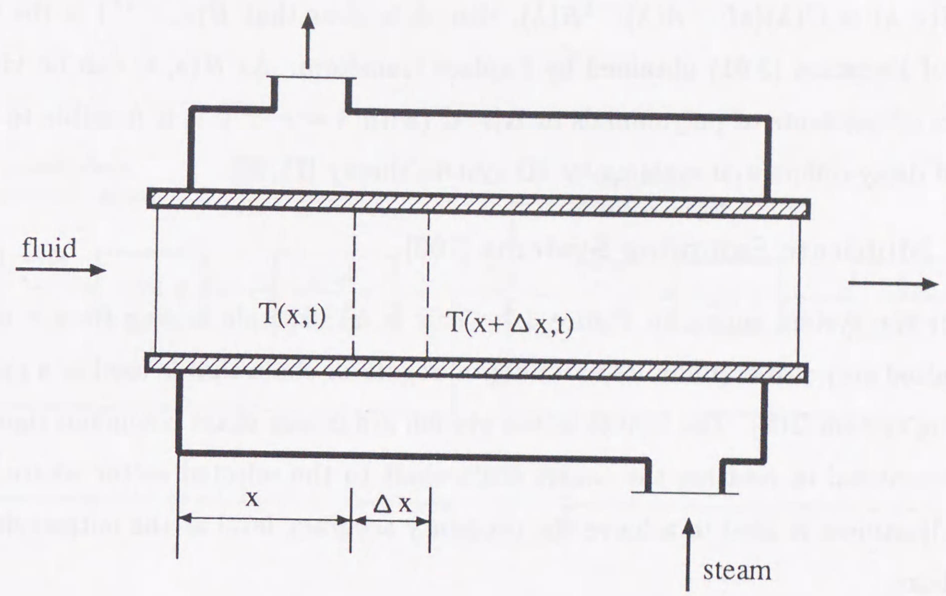


Figure 3.2: Heat exchanger

We can also obtain the transfer function of this system by directly applying 2D z-transformation to Equation (3.56) or from the above state-space model as follows.

$$T(z_1,z_2) = H(z_1,z_2)U(z_1,z_2) \tag{3.59}$$

where

$$H(z_1,z_2) = \frac{bz_2}{1 - a_1 z_2 - a_2 z_1 z_2} \tag{3.60}$$

3.6.2 Delay-differential Systems [75]

Consider the retarded delay-differential system

$$\dot{x}(t) = A(\lambda)x(t) + B(\lambda) \tag{3.61a}$$

$$y(t) = C(\lambda)x(t) \tag{3.61b}$$

where λ is a fixed delay operator of time interval $T > 0$; $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are matrices over $\mathbf{R}[\lambda]$; $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ are real vectors of appropriate dimensions. So the dynamic model (3.61) describes a system with fixed commensurate delays, the duration of each delay being an integral multiple of T.

If $H(s,\lambda) \equiv C(\lambda)(sI - A(\lambda))^{-1}B(\lambda)$, then it is clear that $H(s, e^{-sT})$ is the transfer matrix of Equation (3.61) obtained by Laplace transform. As $H(s,\lambda)$ can be viewed as a matrix of quotients of polynomials in $\mathbf{R}[s,\lambda]$ (with $\lambda = e^{-sT}$), it is possible to analyze retarded delay-differential systems by 2D system theory [75, 90].

3.6.3 Multirate Sampling Systems [106]

Consider the system shown in Figure 3.3. This is an example arising from a model of fine (limited arc)-coarse (unlimited arc) digital regulator chain that is used in a projectile-launching system [106]. The inputs to the system are coarse chain command signals that are instrumental in rotating the coarse chain shaft to the selected sector where the fine chain adjustment is used to achieve the necessary accuracy level at the output shaft.

Defining

$$z_1 = e^{10s}, \qquad z_2 = e^{s} \tag{3.62}$$

and then applying z-transformation to the system, we get the transformed system as in Figure 3.4, which describes the 2D dynamics in the system and can be analyzed by using 2D system theory (see [106] for detail).

3.6.4 Iterative Learning Control Systems [44]

Consider a dynamic system with a linear discrete-time model

$$y(i+1) = Ay(i) + Bu(i) \tag{3.63}$$

where $y \in \mathbf{R}^n$, $u \in \mathbf{R}^r$ are the output and input vectors, respectively. The task of the system is to execute the repetitive operation of tracking a desired output trajectory $y_d(i)$ over a finite discrete-time interval $i \in [0, N]$.

The basic idea of the iterative learning control system methodology is to take advantage of the repetitive feature of system operations in determining the control sequence $u(i)$ such that the tracking performance of the system output will improve with the number of iterations (see Figure 3.5).

It is easy to see that in iterative learning control systems there exist independently two dynamic processes: one represents the dynamics of the controlled plant associated with the evolution of the temporal variable i, and the other reflects the learning iteration and performance improvement in terms of the change of the variable j [44]. Therefore,

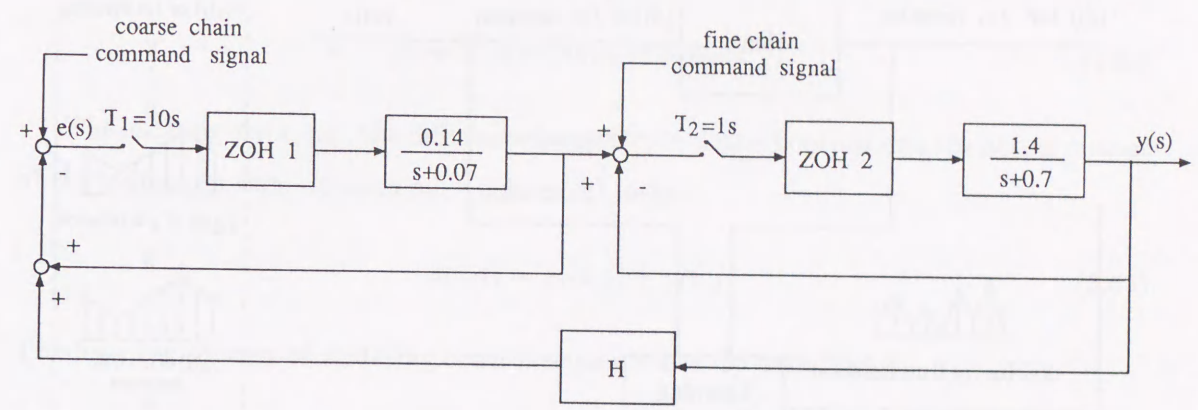


Figure 3.3: Fine-coarse regulator block representation

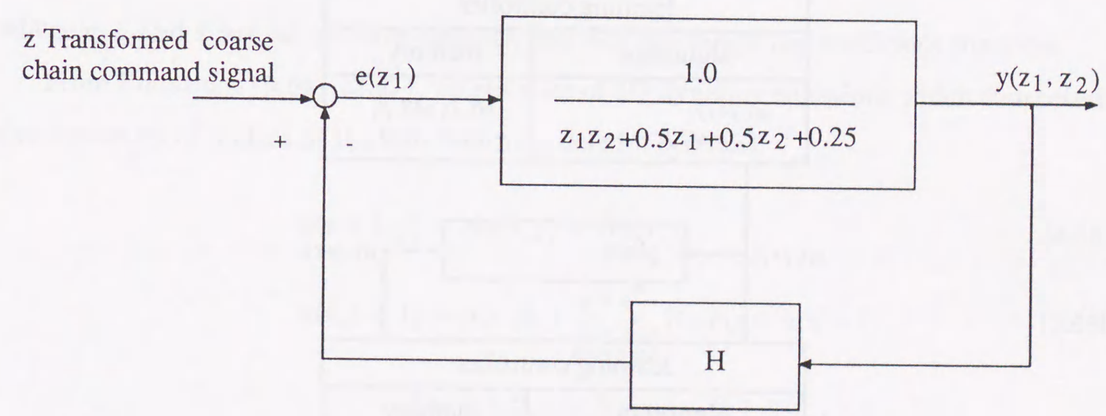


Figure 3.4: z-transformed single-input fine-coarse regulator model

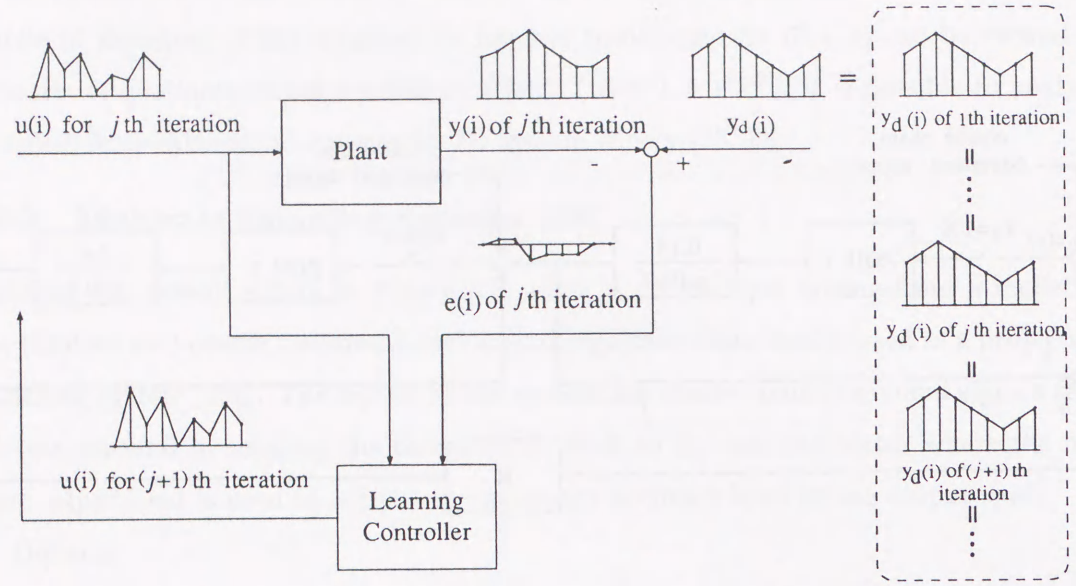


Figure 3.5: Description for iterative learning control systems

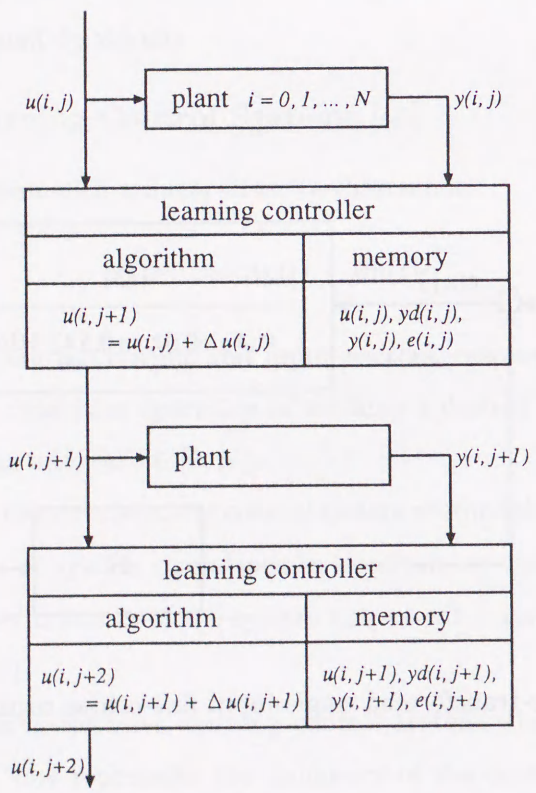


Figure 3.6: The 2D structure of an iterative learning control system

an iterative learning control system can naturally be described by 2D system model. Denoting, by $y(i,j)$ and $u(i,j)$, the output and control sequences of the system in the ith time interval of the jth learning iteration, respectively, we can express Equation (3.63) as

$$y(i+1,j) = Ay(i,j) + Bu(i,j) \tag{3.64}$$

The learning error, i.e., the difference between the desired output and the actual output of the system, is then denoted as

$$e(i,j) = y_d(i,j) - y(i,j) \tag{3.65}$$

The learning process of updating control sequence can be described by

$$u(i,j+1) = u(i,j) + \Delta u(i,j) \tag{3.66}$$

where $\Delta u(i,j)$ is the modification variable of the control sequence in the ith time interval of the jth iteration, reflecting the learning process of the system. Specifically, the value of $\Delta u(i,j)$ must be a function of the error $e(i,j)$, and in the linear case $\Delta u(i,j)$ can be represented in the general form

$$\Delta u(i,j) = \sum_{h=p}^{-q}\sum_{l=0}^{k} K_{hl}e(i-h,j-l) \tag{3.67}$$

where p, q and k are all positive integers and K_{hl} are weighting coefficient matrices.

From Equations (3.64)–(3.67), we get a set of 2D dynamic equations which characterize the dynamics of a class of iterative learning control systems:

$$y(i+1,j) = Ay(i,j) + Bu(i,j) \tag{3.68a}$$

$$u(i,j+1) = u(i,j) + \sum_{h=p}^{-q}\sum_{l=0}^{k} K_{hl}e(i-h,j-l) \tag{3.68b}$$

$$i = 0,1,2,\ldots,N, \quad j = 0,1,2,\ldots$$

The initial conditions of Equation (3.68) are

$$y(0,j) = y_j, \qquad u(i,0) = u_i \tag{3.69}$$

Figure 3.6 shows the structure of an iterative learning control system in 2D notation.

3.6.5 Multipass Processes [83]

Multipass processes are a class of dynamic systems characterised by a series of repetitive operations, over a finite pass length, with interaction between successive passes. In particular, the previous $M \geq 1$ pass outputs act as forcing functions on, and hence influence, the dynamics of the current pass. If $M = 1$ the process is said to be unit memory and otherwise non-unit memory.

Such multipass processes can be illustrated by consideration of machining operations where the material or workpiece involved is processed by a sequence of passes of the processing tool. Some industrial examples, such as long-wall coal cutting and the rolling of metal strips, have been shown in the literature (see, e.g., [82, 83] for detail).

The unique control problem arises in multipass processes is the possible existence of oscillations in the output sequence which increase in amplitude from pass to pass. It has been indicated that this is a problem beyond the scope of existing (1D) linear system theory [82].

The class of discrete non-unit memory linear multipass processes can be described by the state-space model [83]

$$x_k(i+1) = Ax_k(i) + Bu_k(i) + \sum_{j=1}^{M} B_{j-1}y_{k-j}(i) \tag{3.70a}$$

$$y_k(i) = Cx_k(i) + D_0u_k(i) + \sum_{j=1}^{M} D_jy_{k-j}(i) \tag{3.70b}$$

$$x_k(i) \in R^n, \quad y_k(i) \in R^m, \quad u_k(i) \in R^l$$
$$0 \leq i \leq N, \quad x_k(0) = d_k, \quad k \geq 0$$

Denoting $x_k(i)$, $y_k(i)$ and $u_k(i)$ as $x(i,k)$, $y(i,k)$ and $u(i,k)$ respectively, we can rewrite Equation (3.70) in 2D expression as

$$x(i+1,k) = Ax(i,k) + Bu(i,k) + \sum_{j=1}^{M} B_{j-1}y(i,k-j) \tag{3.71a}$$

$$y(i,k) = Cx(i,k) + D_0u(i,k) + \sum_{j=1}^{M} D_jy(i,k-j) \tag{3.71b}$$

$$x(i,k) \in R^n, \quad y(i,k) \in R^m, \quad u(i,k) \in R^l$$
$$0 \leq i \leq N, \quad x(0,k) = d_k, \quad k \geq 0$$

Then, using 2D z-transformation, we can get the transfer matrix of the system (3.71) as follows.

$$Y(z_1,z_2) = H(z_1,z_2)U(z_1,z_2) \tag{3.72}$$

where

$$H(z_1,z_2) = (I_m - \sum_{j=1}^{M} H_j(z_1)z_2^j)^{-1}H_0(z_1)U(z_1,z_2) \tag{3.73}$$

$$H_0(z_1) = C(I_n - z_1A)^{-1}z_1B + D_0 \tag{3.74}$$

$$H_j(z_1) = C(I_n - z_1A)^{-1}z_1B_{j-1} + D_j \tag{3.75}$$

Chapter 4

Output Feedback Stabilizability and Stabilization Algorithms for 2D Systems

4.1 Introduction

The synthesis problem of stabilizing compensators for 2D linear feedback systems has been recently investigated by a number of researchers (see, e.g., [10–15, 36, 49, 68, 79] and the references therein). As in the 1D case, the synthesis procedure can be reduced to solving certain linear equations for polynomial or polynomial matrices in two variables. In particular, it has been shown [12, 49, 68] that a causal 2D multivariable plant given by a (left) matrix fraction description (MFD) $D^{-1}(v,w)N(v,w)$, with $D(v,w)$, $N(v,w) \in \mathbf{M}(\mathbf{R}[v,w])$ and $\det D(0,0) \neq 0$, is output feedback (structurally) stabilizable by a causal 2D compensator $Y(v,w)X^{-1}(v,w)$ if and only if the equation

$$D(v,w)X(v,w) + N(v,w)Y(v,w) = \Phi(v,w) \tag{4.1}$$

holds, where $X(v,w), Y(v,w), \Phi(v,w) \in \mathbf{M}(\mathbf{R}[v,w])$, $\det X(0,0) \neq 0$, and $\det \Phi(v,w) \neq 0$ for any $(v,w) \in \bar{U}^2$. Once Equation (4.1) is solved, the class of all stabilizing compensators can be explicitly parametrized [49]. As a particular aspect of 2D systems, however, Equation (4.1) is not in general solvable for an arbitrarily assigned $\Phi(v,w)$ even when $D(v,w)$ and $N(v,w)$ are (left) factor coprime. In fact the closed-loop characteristic polynomial $\det \Phi(v,w)$ has to vanish at the common zeros of all the maximal order minors $a_i(v,w)$, $i = 1, \ldots, \beta$, of the matrix $[D(v,w)\ N(v,w)]$. Denote by $\mathcal{V}(\mathcal{I})$ the variety of the ideal $\mathcal{I}$ generated by $a_i(v,w)$, $i = 1, \ldots, \beta$. In consequence of [49] and [12], then, a necessary and sufficient condition for the system to be stabilizable, or equivalently for Equation (4.1) to be solvable, is that $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$, namely, $a_i(v,w)$, $i = 1, \ldots, \beta$, share no common zeros in $\bar{U}^2$.

Several problems are naturally associated with the synthesis of 2D compensators. There is, first of all, the problem of testing the feedback stabilizability of a given system without explicit computation of $\mathcal{V}(\mathcal{I})$ or direct solution of Equation (4.1). Then arise the problem of constructing a suitable stable polynomial matrix $\Phi(v,w)$ such that Equation (4.1) holds for some $X(v,w)$ and $Y(v,w)$, and the problem of computing such solution $X(v,w)$, $Y(v,w)$.

With a view to solving the first problem, Zhang [125] has presented a test for SISO 2D systems. However, a basic fact was ignored there that even though the numerator $n(v,w)$ and the denominator $d(v,w)$ of a 2D system transfer function $n(v,w)/d(v,w)$ share some common zeros, say (v_i, w_i), (v_j, w_j), with $|v_i| < 1$ and/or $|w_j| < 1$, it cannot be concluded that these common zeros belong to $\bar{U}^2$. In fact, these zeros lie outside $\bar{U}^2$ whenever the conditions $|v_i| < 1$ and/or $|w_j| < 1$, and $|w_i| > 1$ and/or $|v_j| > 1$, are satisfied simultaneously. Fornasini [36] has recently shown a criterion, in terms of structural properties of a pair of commuting linear transformation, for feedback stabilizability of 2D MIMO systems. While this criterion did not directly provide an efficient algorithm, Bisiacco *et al.* [13] later obtained some further results and developed a linear test algorithm. By this method, the test can be reduced to the solution of a finite family of Lyapunov equations and thus can be accomplished in a finite number of steps.

As for construction of a stable $\Phi(v,w)$ and solution of Equation (4.1), roughly speaking, two kinds of procedures have been developed up to now.

The first kind is due to the researches of Guiver and Bose [49], Bisiacco *et al.* [10–15] and Fornasini [36] which can be recalled as follows.

(i) Construct a stable 2D polynomial, say $s(v,w)$, which vanishes on the variety $\mathcal{V}(\mathcal{I})$.

(ii) According to Hilbert's Nullstellensatz, then, there exist $x_i(v,w) \in \mathbf{R}[v,w]$, $i = 1, \ldots, \beta$, and some integer r such that

$$\sum_{i=1}^{\beta} a_i(v,w) x_i(v,w) = s^r(v,w) \tag{4.2}$$

above-mentioned r, are required. Nevertheless, a $\Phi(v,w)$ having the general form as show in [68] can be obtained, and it is possible to generalize the algorithm to nD ($n > 2$) cases without essential difficulty. Section 4.4 is devoted to the problems of generalization on some concepts of coprimeness and the relevant MFD's. More precisely, the concepts of factor coprimeness and zero coprimeness in the 2D polynomial ring are generalized to the ring of causal Ω-stable (see Section 4.4 for the definition) 2D rational functions as factor Ω-coprimeness and zero Ω-coprimeness, respectively. The corresponding necessary and sufficient condition for the existence of zero Ω-coprime MFD is then derived based on the results obtained in Section 4.3. Moreover, since the defined ring of causal Ω-stable 2D rational functions contains the 2D polynomial ring as a subring, it offers us possibility and convenience to solve the asymptotic and deadbeat servo control problems in a unified way as will be clarified in Chapter 5. Section 4.5 is addressed to the problems of stabilization of 2D systems and parametrization of all stabilizing compensators. Finally, Section 4.6 provides an illustrative example for the proposed methods.

4.2 Stabilizability and Closed-Loop Stable Polynomials

Consider an MIMO 2D linear system given by a left MFD $D^{-1}(v,w)N(v,w)$. Without loss of generality, we suppose that $D(v,w)$ and $N(v,w)$ are left factor coprime. In fact, if $D(v,w)$ and $N(v,w)$ are not left factor coprime, a left greatest common factor, say $R(v,w)$, can be constructively computed and removed [48, 67, 74], and the stability of $\det R(v,w)$ can be checked (see, e.g., [50] or Chapter 3). By the results of Corollary 2.1, then, we always have $X_1(v,w)$, $Y_1(v,w)$, $X_2(v,w)$ and $Y_2(v,w) \in \mathbf{M}(\mathbf{R}[v,w])$ such that

$$D(v,w)X_1(v,w) + N(v,w)Y_1(v,w) = \Phi_1(v) \tag{4.5}$$

$$D(v,w)X_2(v,w) + N(v,w)Y_2(v,w) = \Phi_2(w) \tag{4.6}$$

where $\Phi_1(v) \in \mathbf{M}(\mathbf{R}[v])$, $\Phi_2(w) \in \mathbf{M}(\mathbf{R}[w])$ are diagonal 1D polynomial matrices with non-zero determinants [74].

By employing the long division method, a 1D polynomial can be decomposed, without explicit computation of its roots, into a product of a stable polynomial and a polynomial with all its zeros being unstable [69, 78]. We can, therefore, carry out the decompositions

$$\Phi_1(v) = \Phi_{1u}(v)\Phi_{1s}(v) \tag{4.7}$$

(4.3) may result in a solution $X(v,w)$, $Y(v,w)$ with relatively high degree in v and w [68].

Nevertheless, this procedure is attractive for the advantage that it is easy to be extended to nD ($n > 2$) cases. As a matter of fact, since Gröbner basis approach is generally applicable for multivariable polynomials, the procedure applies directly to nD ($n > 2$) cases as long as a suitable stable nD polynomial can be constructed.

In contrast with the first kind of procedure discussed above, another kind of procedure was initiated by Raman and Liu [79] for SISO 2D systems and later developed by Lin [68] to MIMO case. By the method of [68], a stable $\Phi(v,w)$ and a solution $X(v,w)$, $Y(v,w)$ can be constructed simultaneously without calculating the minors and their Gröbner basis. Further, the obtained $\Phi(v,w)$ is in a more general form which may lead to solution $X(v,w)$, $Y(v,w)$ having less degree in v and w than the solution obtained by the first kind of procedure [68].

According to this procedure, however, one has to compute explicitly all the (unstable) roots of the entries of a diagonal 1D polynomials matrix, say $Q(v)$; and then, for every such root in every polynomial entry of $Q(v)$, successively perform an elimination procedure that involves in general Smith form transformation of certain 1D matrices. Therefore, it is not difficult to see that the procedure may be computationally required when the number of such roots and/or the dimension of $Q(v)$ are relatively large. Furthermore, this approach depends heavily on the technique to reduce the problem to 1D case by explicitly evaluating one of the two variables so that it is rather difficult, if not impossible, for extending to nD ($n > 2$) cases.

The main contents of this chapter are as follows. In Section 4.2, we propose alternative methods for test of output feedback stabilizability and construction of a closed-loop stable polynomial $s(v,w)$ for a given 2D system. By these methods, the considered problems can be entirely reduced to 1D case, and can be solved by using 1D algorithms or Gröbner basis approach. In Section 4.3, then, the "Rabinowitsch trick" mentioned earlier will be extended in some senses to the case of modules over polynomial ring. Based on these results, two new solution algorithms for Equation (4.1) are presented. In particular, one of the algorithms shows that Equation (4.1) over the ring $\mathbf{R}[v,w]$ can be solved through the solution of an equivalent Bezout equation over a polynomial over-ring of $\mathbf{R}[v,w]$ by using the Gröbner basis approach for modules. According to this algorithm, neither computation of any minors or zeros nor estimation of any degrees, such as the

(iv) *For $i = 1,\ldots,n$, $\vec{e}_i$ is an element of the Gröbner basis of the module generated by*

$$\{\vec{f}_1,\ \cdots,\ \vec{f}_{m+n},\ \begin{bmatrix} 0\\ \vdots\\ 0\\ \det\Phi_{1u}(v)\\ 0\\ \vdots\\ 0\end{bmatrix}*,\ \begin{bmatrix} 0\\ \vdots\\ 0\\ \det\Phi_{2u}(w)\\ 0\\ \vdots\\ 0\end{bmatrix}*\} \tag{4.13}$$

where $\vec{e}_i$ denotes the n-tuple having 1 as the element at the ith position and zeros at the other positions, and $*$ denotes the ith position of the related tuples.

Proof.

(i) $\Leftrightarrow$ (ii): As mentioned in the introduction, the output feedback stabilizability of 2D system $D^{-1}(v,w)N(v,w)$ is equivalent to the solvability of Equation (4.1). By the Cauchy-Binet theorem and Equations (4.5) and (4.6), it is clear that if (v_0,w_0) is a common zero of $a_i(v,w)$, $i = 1,\ldots,\beta$, namely, $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$, then $\det\Phi_1(v_0) = 0$ and $\det\Phi_2(w_0) = 0$ simultaneously. This fact obviously implies that

$$\mathcal{V}(\mathcal{I}) \subset \Gamma\{\det\Phi_1\} \cap \Gamma\{\det\Phi_2\} \tag{4.14}$$

On the other hand, it has been mentioned that

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset \tag{4.15}$$

is a necessary and sufficient condition for (i) to be true. By Equation (4.14) and the definitions of $\det\Phi_{1u}(v)$ and $\det\Phi_{2u}(w)$, it is easy to see

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 \subset \Gamma\{\det\Phi_{1u}\} \cap \Gamma\{\det\Phi_{2u}\} \subset \bar{U}^2 \tag{4.16}$$

In view of this fact and Equation (4.15), then, one can readily conclude that (i), or equivalently, Equation (4.1) holds if and only if

$$\mathcal{V}(\mathcal{I}) \cap \{\ \Gamma\{\det\Phi_{1u}\} \cap \Gamma\{\det\Phi_{2u}\}\ \} = \emptyset, \tag{4.17}$$

or equivalently, (ii) is true.

(ii) $\Leftrightarrow$ (iii): As a matter of fact, Equation (4.17) implies, and is also implied by, the zero coprimeness of the polynomials $\det\Phi_{1u}(v)$, $\det\Phi_{2u}(w)$ and $a_i(v,w)$, $i = 1,\ldots,\beta$. By

$$\Phi_2(w) = \Phi_{2u}(w)\Phi_{2s}(w) \tag{4.8}$$

or alternatively,

$$\det \Phi_1(v) = \det \Phi_{1u}(v) \det \Phi_{1s}(v) \tag{4.9}$$

$$\det \Phi_2(w) = \det \Phi_{2u}(w) \det \Phi_{2s}(w) \tag{4.10}$$

such that all the zeros of $\det \Phi_{1u}(\xi)$ and $\det \Phi_{2u}(\xi)$ lie in $\bar{U}$, and all the zeros of $\det \Phi_{1s}(\xi)$ and $\det \Phi_{1s}(\xi)$ lie outside $\bar{U}$.

For a 1D polynomial $f(v) \in \mathbf{R}[v]$, we define the notation $\Gamma\{f(v)\} = \{(v,w) \in \mathbf{C}^2 \mid f(v) = 0\}$. The definition applies analogously to w as well. Then the following results can be given.

Theorem 4.1 *For given left factor coprime MFD $D^{-1}(v,w)N(v,w)$ with $D(v,w) \in \mathbf{R}[v,w]^{n\times n}$, and $N(v,w) \in \mathbf{R}[v,w]^{n\times m}$, define*

$$\begin{aligned} F(v,w) &= \begin{bmatrix} D(v,w) & N(v,w) \end{bmatrix} \\ &= \begin{bmatrix} f_{1,1} & \cdots & f_{1,m+n} \\ \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,m+n} \end{bmatrix} \\ &= \begin{bmatrix} \vec{f_1} & \cdots & \vec{f}_{m+n} \end{bmatrix} \end{aligned} \tag{4.11}$$

where

$$\vec{f_j} = [f_{1,j}\ f_{2,j}\ \cdots\ f_{n,j}]^T \in \mathbf{R}^n[v,w], \quad j = 1,\ldots,m+n \tag{4.12}$$

and denote by $\mathcal{V}(\mathcal{I})$ the variety of the ideal $\mathcal{I}$ generated by all the nth-order minors of $F(v,w)$, i.e., $a_i(v,w)$, $i = 1,\ldots,\beta$, $\beta = (m+n)!/(m!n!)$.

Then the following statements are equivalent:

(i) *The 2D system given by $D^{-1}(v,w)N(v,w)$ is output feedback stabilizable;*

(ii) *For any $(v_0,w_0) \in \Gamma\{\det \Phi_{1u}\} \cap \Gamma\{\det \Phi_{2u}\}$, the matrix $[D(v_0,w_0)\ N(v_0,w_0)]$ is full rank;*

(iii) *A non-zero constant is an element (the only element) included in the Gröbner basis (the reduced Gröbner basis) of the ideal generated by $\det \Phi_{1u}(v)$, $\det \Phi_{2u}(w)$ and $a_i(v,w)$, $i = 1,\ldots,\beta$;*

(iv) *For $i = 1,\ldots,n$, $\vec{e_i}$ is an element of the Gröbner basis of the module generated by*

$$\left\{\vec{f_1},\ \cdots,\ \vec{f}_{m+n},\ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det\Phi_{1u}(v) \\ 0 \\ \vdots \\ 0 \end{bmatrix} *,\ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det\Phi_{2u}(w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} *\right\} \tag{4.13}$$

where $\vec{e_i}$ denotes the n-tuple having 1 as the element at the ith position and zeros at the other positions, and $*$ denotes the ith position of the related tuples.

Proof.

(i) $\Leftrightarrow$ (ii): As mentioned in the introduction, the output feedback stabilizability of 2D system $D^{-1}(v,w)N(v,w)$ is equivalent to the solvability of Equation (4.1). By the Cauchy-Binet theorem and Equations (4.5) and (4.6), it is clear that if (v_0,w_0) is a common zero of $a_i(v,w)$, $i = 1,\ldots,\beta$, namely, $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$, then $\det\Phi_1(v_0) = 0$ and $\det\Phi_2(w_0) = 0$ simultaneously. This fact obviously implies that

$$\mathcal{V}(\mathcal{I}) \subset \Gamma\{\det\Phi_1\} \cap \Gamma\{\det\Phi_2\} \tag{4.14}$$

On the other hand, it has been mentioned that

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset \tag{4.15}$$

is a necessary and sufficient condition for (i) to be true. By Equation (4.14) and the definitions of $\det\Phi_{1u}(v)$ and $\det\Phi_{2u}(w)$, it is easy to see

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 \subset \Gamma\{\det\Phi_{1u}\} \cap \Gamma\{\det\Phi_{2u}\} \subset \bar{U}^2 \tag{4.16}$$

In view of this fact and Equation (4.15), then, one can readily conclude that (i), or equivalently, Equation (4.1) holds if and only if

$$\mathcal{V}(\mathcal{I}) \cap \{\ \Gamma\{\det\Phi_{1u}\} \cap \Gamma\{\det\Phi_{2u}\}\ \} = \emptyset, \tag{4.17}$$

or equivalently, (ii) is true.

(ii) $\Leftrightarrow$ (iii): As a matter of fact, Equation (4.17) implies, and is also implied by, the zero coprimeness of the polynomials $\det\Phi_{1u}(v)$, $\det\Phi_{2u}(w)$ and $a_i(v,w)$, $i = 1,\ldots,\beta$. By

Theorem 2.2, this is equivalent to the equation

$$\sum_{i=1}^{\beta} a_i(v,w)\tilde{x}_i + \bar{x}_1(v,w)\det\Phi_{1u}(v) + \bar{x}_2(v,w)\det\Phi_{2u}(w) = 1. \tag{4.18}$$

where $\tilde{x}_i$, $\bar{x}_j \in \mathbf{R}[v,w]$, $i = 1,\ldots,\beta$, $j = 1,2$. According to the properties of Gröbner basis stated in Theorem 2.6, this equation is solvable if and only if 1 can be reduced to zero with respect to the Gröbner basis of the ideal generated by $\det\Phi_{1u}(v)$, $\det\Phi_{2u}(w)$ and $a_i(v,w)$, $i = 1,\ldots,\beta$. However, this is true if and only if (iii) holds [23].

(iii) $\Leftrightarrow$ (iv): Suppose that (iii) holds true, then we have the result of Equation (4.18). By the results (of Lemmas 4.1, 4.2) which will be shown in the next section, we have that if Equation (4.18) holds, then we can obtain polynomial $z_{1,i}(v,w),\ldots,z_{m+n,i}(v,w)$ and $\bar{x}_{1,i}(v,w)$, $\bar{x}_{1,i}(v,w)$ such that

$$z_{1,i}(v,w)\vec{f}_1(v,w) + \cdots + z_{m+n,i}(v,w)\vec{f}_{m+n}(v,w) +$$
$$\bar{x}_{1,i}(v,w)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det\Phi_{1u}(v) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * + \bar{x}_{2,i}(v,w)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det\Phi_{2u}(w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} *, \ i = 1,\ldots,n \tag{4.19}$$

In view of the properties of Gröbner basis for module (see Theorem 2.7), the claim of (iv) is concluded.

Conversely, if (iv) is true, then by using Gröbner approach for module we can find $z_{1,i}(v,w),\ldots,z_{m+n,i}(v,w)$, and $\bar{x}_{1,i}$, $\bar{x}_{2,i}$, $i = 1,\ldots,n$, such that Equation (4.19) is satisfied, which can be written in the matrix form as

$$F(v,w)Z(v,w) = \Phi(v,w) = \begin{bmatrix} \phi_{1,1}(v,w) & 0 & \cdots & 0 \\ 0 & \phi_{2,2}(v,w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{n,n}(v,w) \end{bmatrix} \tag{4.20}$$

where

$$\phi_{i,i}(v,w) = 1 - \bar{x}_{1,i}(v,w)\det\Phi_{1u}(v) - \bar{x}_{2,i}(v,w)\det\Phi_{2u}(w), \qquad i = 1,\ldots,n \tag{4.21}$$

and

$$Z(v,w) = [z_{j,i}(v,w)]^T \in \mathbf{R}^{(m+n)\times n}[v,w],$$
$$i = 1,\ldots,n, \qquad j = 1,\ldots,m+n \tag{4.22}$$

Applying Cauchy-Binet Theorem to Equation (4.20), we get

$$\sum_{k=1}^{\beta} a_k(v,w)\tilde{x}_k(v,w) = \prod_{i=1}^{n}\phi_{i,i}(v,w)$$
$$= \prod_{i=1}^{n}(1 - \bar{x}_{1,i}(v,w)\det\Phi_{1u}(v) - \bar{x}_{2,i}(v,w)\det\Phi_{2u}(w)) \tag{4.23}$$

where $a_k(v,w)$, $\tilde{x}_k(v,w)$ correspond to the $n \times n$ minors of $F(v,w)$ and $Z(v,w)$. Further, expanding the product on the right-hand side of Equation (4.23), we can have

$$\sum_{k=1}^{\beta} a_k(v,w)\tilde{x}_k(v,w) + \bar{x}_1(v,w)\det\Phi_{1u}(v) + \bar{x}_2(v,w)\det\Phi_{2u}(w) = 1 \tag{4.24}$$

for some $\bar{x}_1(v,w)$, $\bar{x}_2(v,w) \in \mathbf{R}[v,w]$. This is obviously identical to Equation (4.18) that implies (iii). □

Theorem 4.2 *Suppose that $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$. Then the polynomial $s(v,w)$ defined as*

$$s(v,w) = \det\Phi_{1s}(v)\det\Phi_{2s}(w) \tag{4.25}$$

vanishes on $\mathcal{V}(\mathcal{I})$ and is stable, namely, devoid of zeros in $\bar{U}^2$.

Proof.

Suppose that $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$. Since $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ by assumption, it follows that

$$|v_0| > 1 \quad \text{and/or} \quad |w_0| > 1. \tag{4.26}$$

By using Equation (4.14) and taking into account the constraint of Equation (4.26), it can be concluded that $\det\Phi_{1s}(v_0) = 0$ and/or $\det\Phi_{2s}(w_0) = 0$. In either case, however,

$$s(v_0,w_0) = \det\Phi_{1s}(v_0)\det\Phi_{2s}(w_0) = 0 \tag{4.27}$$

which shows that $s(v,w)$ vanishes at every $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$. By the definitions of $\det\Phi_{1s}(v)$ and $\det\Phi_{2s}(w)$, it is clear that $s(v,w)$ possesses no zeros in $\bar{U}^2$. □

It is noted that Theorem 4.1 provides three approaches to test the stabilizability. By the statement (ii) of Theorem 4.1, we see that if $\Gamma\{\det \Phi_{1u}\} \cap \Gamma\{\det \Phi_{2u}\}$ is explicitly available, the stabilizability test reduces to checking the rank condition of certain constant matrices. Since the problem has been entirely reduced to the 1D case, it is easy to compute numerically the zeros of $\det \Phi_{1u}(\xi)$ and $\det \Phi_{2u}(\xi)$ by using well-developed 1D algorithms. In general, however, the computation may not be efficient when the degrees of $\det \Phi_{1u}(\xi)$ and $\det \Phi_{2u}(\xi)$ are relatively high. The statement (iii) of Theorem 4.1 shows that by using Gröbner basis approach the stabilizability can be tested without explicit computation of any zeros. As mentioned earlier, however, it would be computationally demanding to calculate all the maximal order minors $a_i(v,w)$, $i = 1, \ldots, \beta$, and the Gröbner basis of $\mathcal{I}$. In contrast with the above two, the statement (iv) reveals that the stabilizability can be effectively tested by directly employing Gröbner basis approach for module, without requiring computation of any zeros or minors.

Additionally, these results give a significant insight to some structural properties of the stabilizability test problem of nD systems. In other words, these results show a possible way to reduce the stabilizability test of nD systems to 1D case without computing any zeros. In fact, based on some results obtained by Gröbner bases [23], the basic ideas adopted in Theorem 4.1 and, in addition, Theorem 4.2 can be directly employed to establish procedures for the stabilizability test and the construction of closed-loop stable polynomial of an nD ($n > 2$) system, provided that the ideal $\mathcal{I}$ is of zero-dimension.

4.3 Solution of Unilateral 2D Polynomial Matrix Equation

The purpose of this section is to develop a solution procedure of Equation (4.1) by applying the Gröbner basis approach for modules over polynomial ring (see, e.g., [43, 73, 105]). Moreover, we expect that the developed procedure should share the advantages, and at the same time, should not suffer from the disadvantages, which we discussed in the introduction for the two kinds of known procedures.

In the following, we first extend the "Rabinowitsch trick" to the case of modules over polynomial ring in the senses of Lemma 4.1 and Lemma 4.2. Then, a general consequence from these results is summarized in Theorem 4.3. Based on these results, we will propose two solution algorithms for Equation (4.1).

Now, we first give the following lemmas.

Lemma 4.1 *Define $D(v,w)$, $N(v,w)$, $F(v,w)$ and $\mathcal{V}(\mathcal{I})$ as in Theorem 4.1. If $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$, then for any fixed $i \in \{1, 2, \ldots, n\}$ there exist $\bar{x}_i(v,w,t_i)$ and $\tilde{x}_{i,j}(v,w,t_i) \in \mathbf{R}[v,w,t_i]$, $j = 1, \ldots, m+n$, such that*

$$\tilde{x}_{i,1}(v,w,t_i)\vec{f}_1(v,w) + \cdots + \tilde{x}_{i,m+n}(v,w,t_i)\vec{f}_{m+n}(v,w) + \bar{x}_i(v,w,t_i)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 - t_i s(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} * \tag{4.28}$$

where t_i are new indeterminates and $s(v,w)$ is a stable 2D polynomial vanishing on $\mathcal{V}(\mathcal{I})$. ($*$ denotes the ith position of the related n-tuples.)

Proof.

It is noted, first of all, that a stable polynomial $s(v,w)$ vanishing on $\mathcal{V}(\mathcal{I})$ can be obtained as in Theorem 2.2 or [13] whenever the condition

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset \tag{4.29}$$

is satisfied.

It is also easy to see that, without loss of generality, we can interchange, for $i = 2, \ldots, n$ respectively, the first element and the ith one of every n-tuple in Equations (4.28), and thus transform the problems when $i = 2, 3, \ldots, n$ to the case of $i = 1$. Therefore, we have just to show the proof for the case $i = 1$. For brevity, the problem to be proved can be reformulated as follows: under the assumption that $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$, to show that there exist $\bar{x}(v,w,t)$ and $\tilde{x}_i(v,w,t) \in \mathbf{R}[v,w,t]$, $i = 1, \ldots, m+n$, such that

$$\tilde{x}_1(v,w,t)\vec{f}_1(v,w) + \cdots + \tilde{x}_{m+n}(v,w,t)\vec{f}_{m+n}(v,w) + \bar{x}(v,w,t)\begin{bmatrix} 1 - ts(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.30}$$

where t is a newly introduced indeterminate and $s(v,w)$ is a stable 2D polynomial vanishing on $\mathcal{V}(\mathcal{I})$.

Now we begin formally to give the proof. Let $\bar{F}(v,w)$ be the matrix formed by removing the first row of $F(v,w)$, i.e.,

$$\tilde{F}(v,w) = \begin{bmatrix} f_{2,1} & \cdots & f_{2,m+n} \\ \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,m+n} \end{bmatrix} \in \mathbf{R}[v,w]^{(n-1)\times(m+n)} \tag{4.31}$$

Since $\det D(v,w) \neq 0$, in the $2 \sim n$ rows of $D(v,w)$ there exists at least one $(n-1)$th-order non-zero minor. Denote by $\tilde{D}(v,w) \in \mathbf{R}[v,w]^{(n-1)\times(n-1)}$ the sub-matrix that corresponds to a non-zero minor, and denote by $\tilde{N}(v,w) \in \mathbf{R}[v,w]^{(n-1)\times(m+1)}$ the sub-matrix formed by eliminating the columns of $\tilde{F}(v,w)$ that have been already included in $\tilde{D}(v,w)$. Without loss of generality, we can assume that $\tilde{D}(v,w)$ is formed by deleting the first row and the last column of $D(v,w)$. Hence, $\tilde{F}(v,w)$ defined in Equation (4.31) can be represented as follows.

$$\tilde{F}(v,w) = \left[\tilde{D}(v,w)\ \tilde{N}(v,w)\right] \tag{4.32}$$

Under the assumption that $D(v,w)$ and $N(v,w)$ are left factor coprime, which is equivalent to the coprimeness of all the nth-order minors of $F(v,w)$ (see Theorem 2.1 or [123]), we show that $\tilde{D}(v,w)$ and $\tilde{N}(v,w)$ are also left factor coprime. By utilizing Laplace expansion formula to the first row of all the square nth-order sub-matrices of $F(v,w)$, which correspond to all the nth-order minors of $F(v,w)$, we can express every nth-order minor of $F(v,w)$ as a linear combination of the $(n-1)$th-order minors of $\tilde{F}(v,w)$. Now, if the $(n-1)$th-order minors of $\tilde{F}(v,w)$ have a nontrivial common factor, which implies that $\tilde{D}(v,w)$ and $\tilde{N}(v,w)$ are not left factor coprime, then the nth-order minors of $F(v,w)$ share the same factor. This is obviously a contradiction to the assumption. Thus we have that $\tilde{D}(v,w)$ and $\tilde{N}(v,w)$ must be left factor coprime.

We can then find right factor coprime matrices $\bar{D}(v,w) \in \mathbf{R}[v,w]^{(m+1)\times(m+1)}$ and $\bar{N}(v,w) \in \mathbf{R}[v,w]^{(n-1)\times(m+1)}$ such that

$$\tilde{F}(v,w)\bar{F}(v,w) = \left[\tilde{D}(v,w)\ \tilde{N}(v,w)\right] \begin{bmatrix} -\bar{N}(v,w) \\ \bar{D}(v,w) \end{bmatrix} = 0 \tag{4.33}$$

where

$$\bar{F}(v,w) = \begin{bmatrix} -\bar{N}(v,w) \\ \bar{D}(v,w) \end{bmatrix} \in \mathbf{R}[v,w]^{(m+n)\times(m+1)} \tag{4.34}$$

Let $\tilde{\mathcal{I}}$ and $\bar{\mathcal{I}}$ be the ideal generated by all the $(n-1)$th-order minors of $\tilde{F}(v,w)$ and the $(m+1)$th-order minors of $\bar{F}(v,w)$, and $\mathcal{V}(\tilde{\mathcal{I}})$, $\mathcal{V}(\bar{\mathcal{I}})$ be the varieties of $\tilde{\mathcal{I}}$ and $\bar{\mathcal{I}}$,

respectively. According to the results of Theorem 2.4 (see also [12, 15]), then, we have

$$\mathcal{V}(\tilde{\mathcal{I}}) = \mathcal{V}(\bar{\mathcal{I}}) \tag{4.35}$$

In light of Equation (4.35) and the fact that every nth-order minor of $F(v,w)$ can be expressed as a linear combination of the $(n-1)$th-order minors of $\tilde{F}(v,w)$, it is direct to see that

$$\mathcal{V}(\bar{\mathcal{I}}) \subset \mathcal{V}(\mathcal{I}). \tag{4.36}$$

Let

$$\begin{aligned} \bar{F}^T(v,w) &= \left[-\bar{N}^T(v,w)\ \bar{D}^T(v,w)\right] \\ &= \begin{bmatrix} \bar{f}_{1,1} & \cdots & \bar{f}_{1,m+n} \\ \vdots & & \vdots \\ \bar{f}_{m+1,1} & \cdots & \bar{f}_{m+1,m+n} \end{bmatrix} \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} \tilde{F}(v,w) &= \left[\tilde{D}(v,w)\ \tilde{N}(v,w)\right] \\ &= \left[\vec{\tilde{f}}_1 \ \cdots\ \vec{\tilde{f}}_{m+n}\right] \end{aligned} \tag{4.38}$$

where

$$\vec{\tilde{f}}_j = [f_{2,j}\ \ f_{3,j}\ \ \cdots\ \ f_{n,j}]^T \in \mathbf{R}[v,w]^{n-1}, \quad j = 1,\ldots,m+n \tag{4.39}$$

By Equation (4.33), then, we see that the rows of $\bar{F}^T(v,w)$ are a basis for the module of syzygies of $\vec{\tilde{f}}_1, \ldots, \vec{\tilde{f}}_{m+n}$, i.e.,

$$\bar{f}_{i,1}\vec{\tilde{f}}_1 + \bar{f}_{i,2}\vec{\tilde{f}}_2 + \ldots + \bar{f}_{i,m+n}\vec{\tilde{f}}_{m+n} = \vec{0}, \qquad i = 1,\ldots,m+1 \tag{4.40}$$

Further, considering

$$y_0\vec{0} + y_1\vec{\tilde{f}}_1 + \ldots + y_{m+n}\vec{\tilde{f}}_{m+n} = \vec{0} \tag{4.41}$$

we can construct the matrix

$$\bar{F}'(v,w) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \bar{F}^T(v,w) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \bar{f}_{1,1} & \cdots & \bar{f}_{1,m+n} \\ \vdots & & & \\ 0 & \bar{f}_{m+1,1} & \cdots & \bar{f}_{m+1,m+n} \end{bmatrix} \in \mathbf{R}[v,w]^{(m+2)\times(m+n+1)} \tag{4.42}$$

such that the rows of $\bar{F}'(v,w)$ are a basis for the module of syzygies of $\vec{0}, \vec{\bar{f}}_1, \ldots, \vec{\bar{f}}_{m+n}$.

Next, define

$$\bar{F}'(v,w) \begin{bmatrix} 1-ts \\ f_{1,1} \\ \vdots \\ f_{1,m+n} \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{m+1} \end{bmatrix} \in \mathbf{R}[v,w]^{m+2} \tag{4.43}$$

where

$$g_0 = 1 - ts \tag{4.44}$$

$$\begin{bmatrix} g_1 \\ \vdots \\ g_{m+1} \end{bmatrix} = \bar{F}^T(v,w) \begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m+n} \end{bmatrix} \tag{4.45}$$

Following the results of [107] (see also [105]), if we can find some $p_0, p_1, \ldots, p_{m+1} \in \mathbf{R}[v,w]$ such that

$$p_0 g_0 + p_1 g_1 + \cdots + p_{m+1} g_{m+1} = 1 \tag{4.46}$$

or equivalently,

$$p_0(1-ts) + p_1 g_1 + \cdots + p_{m+1} g_{m+1} = 1 \tag{4.47}$$

then we will have $\vec{p} \cdot \bar{F}'$ with

$$\vec{p} = [p_0 \ p_1 \ \cdots \ p_{m+1}] \tag{4.48}$$

as a solution for Equation (4.30). Namely, for all $i = 2, 3, \ldots, n$,

$$\underbrace{\vec{p} \cdot \bar{F}'(v,w) \begin{bmatrix} 0 \\ f_{i,1} \\ \vdots \\ f_{i,m+n} \end{bmatrix}}_{=0} = 0 \tag{4.49}$$

and (when $i = 1$),

$$\vec{p} \cdot \bar{F}'(v,w) \begin{bmatrix} 1-ts \\ f_{1,1} \\ \vdots \\ f_{1,m+n} \end{bmatrix} = \vec{p} \cdot \begin{bmatrix} 1-ts \\ g_1 \\ \vdots \\ g_{m+1} \end{bmatrix} = 1 \tag{4.50}$$

To complete the proof, therefore, what we have to do now is just to show Equation (4.47) is solvable. A possible way to show the solvability of Equation (4.47) is to see whether or not $s(v,w)$ vanishes on the common zeros of $g_1, \ldots, g_{m+1}$, namely, whether or not

$$g_1(v_0,w_0) = g_2(v_0,w_0) = \cdots = g_{m+1}(v_0,w_0) = 0 \tag{4.51}$$

implies

$$s(v_0,w_0) = 0. \tag{4.52}$$

If this is true, $g_1, \ldots, g_{m+1}$, $(1-ts)$ share no common zeros so that Equation (4.47) can be constructively solved by, for example, the Gröbner basis approach [23] or the methods proposed in [7, 8] and [79].

Now we show that this is indeed true. Suppose that the condition of Equation (4.51) is satisfied. Then from Equation (4.45), it is implied that

$$f_{1,1}(v_0,w_0) = \cdots = f_{1,m+n}(v_0,w_0) = 0 \tag{4.53}$$

and/or

$$\operatorname{rank} \bar{F}^T(v_0,w_0) < m+1 \tag{4.54}$$

or in other words, all the $(m+1)$th-order minors of $\bar{F}^T(v_0,w_0)$ are zero. When the condition of Equation (4.53) holds, it is trivial to see that all nth-order minors of $F(v_0,w_0)$ are zero that means $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$. On the other hand, if the condition of Equation (4.54) is true, then by Equation (4.36), it is also concluded that $(v_0,w_0) \in \mathcal{V}(\mathcal{I})$. In view of that $s(v,w)$ vanishes at every point of $\mathcal{V}(\mathcal{I})$, then, we conclude that there exist solution for Equation (4.47).

The whole proof has thus been completed. □

More generally, the following result can be given.

Lemma 4.2 *Define $D(v,w)$, $N(v,w)$, $F(v,w)$ and $\mathcal{V}(\mathcal{I})$ as in Theorem* 4.1. *Let $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ and, in addition,*

$$s(v,w) = s_1(v,w) s_2(v,w) \cdots s_l(v,w) \tag{4.55}$$

be a stable polynomial vanishing on $\mathcal{V}(\mathcal{I})$. Then, for any fixed $i \in \{1,2,\ldots,n\}$, there exist $\bar{x}_{i,j}(v,w,\mathbf{t}_i) \in \mathbf{R}[v,w,\mathbf{t}_i]$, with $\mathbf{t}_i = (t_{i1},\ldots,t_{il})$, $j = 1,\ldots,m+n$, and $\bar{x}_{i,k}(v,w,\mathbf{t}_i) \in$

$\mathbf{R}[v, w, \mathbf{t}_i]$, $k = 1, \ldots, l$, *such that*

$$\tilde{x}_{i,1}(v,w,\mathbf{t}_i)\vec{f}_1(v,w) + \cdots + \tilde{x}_{i,m+n}(v,w,\mathbf{t}_i)\vec{f}_{m+n}(v,w) +$$

$$+\bar{x}_{i,1}(v,w,\mathbf{t}_i)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1-t_{i1}s_1(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix}* + \cdots + \bar{x}_{i,l}(v,w,\mathbf{t}_i)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1-t_{il}s_l(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix}* = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}* \quad (4.56)$$

where t_{ik}, $k = 1, \ldots, l$ *are new indeterminates. (Here, again,* $*$ *denotes the ith position of the related n-tuples.)*

Proof.

It is first noted that if $s(v,w) = s_1(v,w) \cdots s_l(v,w)$ vanishes on $\mathcal{V}(\mathcal{I})$, then for any $(v_0, w_0) \in \mathcal{V}(\mathcal{I})$ we have that

$$s_1(v_0, w_0) = 0 \quad \text{and/or} \quad s_2(v_0, w_0) = 0 \quad \cdots \text{ and/or} \quad s_l(v_0, w_0) = 0 \qquad (4.57)$$

It is now obvious that for v, w and the newly introduced indeterminates $t_1, \cdots, t_l$, the polynomials $a_1(v,w), \cdots, a_\beta(v,w)$ and $1 - t_1 s_1(v,w), \cdots, 1 - t_l s_l(v,w)$ have no common zeros, namely, they are zero coprime on $\mathbf{R}[v, w, \mathbf{t}]$ where $\mathbf{t} = (t_1, \ldots, t_l)$. By Hilbert's Nullstellensatz, then, there exist $\tilde{y}_i(v, w, \mathbf{t})$, $\bar{y}_j(v, w, \mathbf{t}) \in \mathbf{R}[v, w, \mathbf{t}]$, for $i = 1, \ldots, \beta$ and $j = 1, \ldots, l$, such that

$$\sum_{i=1}^{\beta} \tilde{y}_i(v,w,\mathbf{t})a_i(v,w) + \sum_{j=1}^{l} \bar{y}_j(v,w,\mathbf{t})(1 - t_j s_j(v,w)) = 1. \qquad (4.58)$$

By using this fact and noting that Equations (4.42) and (4.43) can be replaced by

$$\bar{F}'(v,w) = \begin{bmatrix} I_l & \mathbf{0} \\ \mathbf{0} & \bar{F}^T(v,w) \end{bmatrix} \in \mathbf{R}[v,w]^{(m+l+1)\times(m+n+l)} \qquad (4.59)$$

where I_l denotes $l \times l$ identity matrix, and

$$\bar{F}'(v,w)\begin{bmatrix} 1-t_1s_1 \\ \vdots \\ 1-t_ls_l \\ f_{1,1} \\ \vdots \\ f_{1,m+n} \end{bmatrix} = \begin{bmatrix} g_{01} \\ \vdots \\ g_{0l} \\ g_1 \\ \vdots \\ g_{m+1} \end{bmatrix} \in \mathbf{R}[v,w]^{m+l+1} \qquad (4.60)$$

where

$$g_{0j} = 1 - t_j s_j, \qquad j = 1, \ldots, l \qquad (4.61)$$

$$\begin{bmatrix} g_1 \\ \vdots \\ g_{m+1} \end{bmatrix} = \bar{F}^T(v,w)\begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,m+n} \end{bmatrix}, \qquad (4.62)$$

we can show the proof in the same way as for Lemma 4.1. □

Based on the results of the above lemmas, the following theorem can be readily established.

Theorem 4.3 *Define* $D(v,w)$, $N(v,w)$, $F(v,w)$ *and* $\mathcal{V}(\mathcal{I})$ *as in Theorem* 4.1. *Then, for all* $i = 1, \ldots, n$, *there exist* $\bar{x}_i(v, w, t_i)$ *and* $\tilde{x}_{i,j}(v, w, t_i) \in \mathbf{R}[v, w, t_i]$, $j = 1, \ldots, m+n$, *such that Equation* (4.28) *holds, or more generally, there exist* $\tilde{x}_{i,j}(v, w, \mathbf{t}_i) \in \mathbf{R}[v, w, \mathbf{t}_i]$, *with* $\mathbf{t}_i = (t_{i1}, \ldots, t_{il})$, $j = 1, \ldots, m+n$, *and* $\bar{x}_{i,k}(v, w, \mathbf{t}_i) \in \mathbf{R}[v, w, \mathbf{t}_i]$, $k = 1, \ldots, l$, *such that Equation* (4.56) *holds, if and only if* $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$.

Proof.

The sufficiency has been shown in Lemmas 4.1 and 4.2, and the necessity can be easily proved as follows. Suppose that Equations (4.28) and (4.56) hold. For $i = 1, \ldots, n$, substituting $t_i = 1/s$ into Equation (4.28), or $t_{ij} = 1/s_j(v,w)$, $j = 1, \ldots, l$, into Equation (4.56) and clearing out all the denominators, we obtain, for every $i = 1, \ldots, n$, a relation in the form of

$$x_{i,1}(v,w)\vec{f}_1(v,w) + \cdots + x_{i,m+n}(v,w)\vec{f}_{m+n}(v,w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s^{r_i}(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix}* \qquad (4.63)$$

where $x_{i,j}(v,w) \in \mathbf{R}[v,w]$, $j = 1,\ldots,m+n$, and r_i are some positive integers, or

$$x_{i,1}(v,w)\vec{f}_1(v,w) + \cdots + x_{i,m+n}(v,w)\vec{f}_{m+n}(v,w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s_1^{r_{i1}}(v,w)\cdots s_l^{r_{il}}(v,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * \tag{4.64}$$

where $x_{i,j}(v,w) \in \mathbf{R}[v,w]$, $j = 1,\ldots,m+n$, and r_{ik}, $k = 1,\ldots,l$ are some positive integers. Since Equations (4.63) and (4.64) hold true for every i, we can obtain a solution $X(v,w)$, $Y(v,w)$ and a stable $\Phi(v,w)$ to Equation (4.1) with

$$[X^T(v,w)\ \ Y^T(x,w)] = [x_{i,j}(v,w)], \quad i = 1,\ldots,n, \quad j = 1\ldots,m+n \tag{4.65}$$

where $x_{i,j}(v,w)$ are the ones obtained in Equation (4.63) or Equation (4.64), and

$$\Phi(v,w) = \begin{bmatrix} s^{r_1}(v,w) & 0 & \cdots & 0 \\ 0 & s^{r_2}(v,w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{r_n}(v,w) \end{bmatrix}, \tag{4.66}$$

or

$$\Phi(v,w) = \begin{bmatrix} s_1^{r_{11}}s_2^{r_{12}}\cdots s_l^{r_{1l}} & 0 & \cdots & 0 \\ 0 & s_1^{r_{21}}s_2^{r_{22}}\cdots s_l^{r_{2l}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_1^{r_{n1}}s_2^{r_{n2}}\cdots s_l^{r_{nl}} \end{bmatrix}. \tag{4.67}$$

Now, the necessity is clear as $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ is a well-known necessary (and sufficient) condition for Equation (4.1) to be solvable (see, e.g., [49]). □

For the convenience of comparison with some previously known results, we would like to rephrase the results obtained in Theorem 4.3 as the following corollary.

Corollary 4.1 *For given left factor coprime MFD $D^{-1}(v,w)N(v,w)$ where $D(v,w) \in \mathbf{R}[v,w]^{n\times n}$, $N(v,w) \in \mathbf{R}[v,w]^{n\times m}$, there exist $X(v,w) \in \mathbf{R}[v,w]^{n\times n}$ and $Y(v,w) \in \mathbf{R}[v,w]^{m\times n}$ such that*

$$D(v,w)X(v,w) + N(v,w)Y(v,w) = \begin{bmatrix} s^{r_1}(v,w) & 0 & \cdots & 0 \\ 0 & s^{r_2}(v,w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{r_n}(v,w) \end{bmatrix}, \tag{4.68}$$

or more generally,

$$D(v,w)X(v,w) + N(v,w)Y(v,w) =$$
$$\begin{bmatrix} s_1^{r_{11}}s_2^{r_{12}}\cdots s_l^{r_{1l}} & 0 & \cdots & 0 \\ 0 & s_1^{r_{21}}s_2^{r_{22}}\cdots s_l^{r_{2l}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_1^{r_{n1}}s_2^{r_{n2}}\cdots s_l^{r_{nl}} \end{bmatrix} \tag{4.69}$$

if and only if

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset \tag{4.70}$$

where

$$s(v,w) = s_1(v,w)s_2(v,w)\cdots s_l(v,w) \tag{4.71}$$

is a stable polynomial vanishing on $\mathcal{V}(\mathcal{I})$, and r_i, r_{ik} are some positive integers for $k = 1,\ldots,l$, $i = 1,\ldots,n$.

Proof. It is obvious from Theorem 4.3. □

By comparing Equations (4.68) and (4.3), first of all, it is easy to see that we have achieved here a less restrictive form for the stable matrix $\Phi(v,w)$ in Equation (4.1). For Equation (4.68) does not require the restriction that the polynomial entries of the diagonal matrix $\Phi(v,w)$ possess a same power. Furthermore, Equation (4.69) shows that when $s(v,w)$ is given by some factorization, all the involved factors may also have different powers. Noting that the power of some factor may be zero, which corresponds to the case when some $\bar{x}_{i,j} = 0$ in Equation (4.56), the form given on the right-hand side of Equation (4.69) is essentially identical to the one obtained in [68]. As mentioned earlier, this form may give solution $X(v,w)$, $Y(v,w)$ to Equation (4.1) with relatively lower degree in v and w. Since an $s(v,w)$ constructed by the method of Theorem 4.2 takes the separated form, i.e., a product of two 1D polynomials in v and w respectively, further decompositions are always possible by 1D approaches whenever one likes to do so.

Moreover, it might be noted that the procedures shown in the proofs of Lemmas 4.1 and 4.2 are constructive, so we can certainly establish a solution algorithm of Equation (4.1) based on the procedures. We remark that, however, the above results are given in a form convenient for the purpose to develop a solution algorithm by directly using Gröbner basis approach for modules over polynomial ring (see, e.g., [43, 73, 105]). In fact, if it is just for the aim of obtaining solution of Equation (4.1), the procedures shown above can be further simplified.

As an alternative method, we summarize the procedures and show the possible simplification by the following algorithm, provided that a stable polynomial $s(v,w)$ vanishing on $\mathcal{V}(\mathcal{I})$ has been obtained by the methods proposed in Section 4.2 or [13].

Algorithm 4.1

Input: $F(v,w) = [D(v,w) \;\; N(v,w)] \in \mathbf{R}[v,w]^{n\times(n+m)}$, a stable polynomial $s(v,w) \in \mathbf{R}[v,w]$ vanishing over $\mathcal{V}(\mathcal{I})$.

Output: $X(v,w)$, $Y(v,w)$ and $\Phi(v,w)$ for Equation (4.1).

For $i = 1,\dots,n$ do step 1 $\sim$ step 6:

step 1. Construct

$$\begin{aligned}\tilde{F}_i(v,w) &= [\tilde{D}_i(v,w)\; \tilde{N}_i(v,w)] \\ &= [\vec{\tilde{f}}_{i,1}\; \vec{\tilde{f}}_{i,2} \cdots \vec{\tilde{f}}_{i,m+n}]\end{aligned} \tag{4.72}$$

by eliminating the ith row of $F(v,w)$.

Note: Here, we suppose that $\det \tilde{D}_i(v,w) \neq 0$. This can be satisfied by proper column permutations of $F(v,w)$ whenever $\det D(v,w) \neq 0$.

step 2. Find

$$\bar{F}_i^T(v,w) = [-\bar{N}_i^T(v,w) \;\; \bar{D}_i^T(v,w)] \tag{4.73}$$

such that $F_i(v,w)\bar{F}_i(v,w) = 0$, and that $\bar{D}_i(v,w)$ and $\bar{N}_i(v,w)$ are right factor coprime.

step 3. Compute

$$[g_{i,1}\; g_{i,2}\; \cdots\; g_{i,m+1}]^T = \bar{F}_i^T [f_{i,1} \;\; f_{i,2} \;\; f_{i,m+n}]^T \tag{4.74}$$

where $[f_{i,1} \;\; f_{i,2} \;\; f_{i,m+n}]$ is the ith row of $F(v,w)$.

step 4. Solve, by Gröbner basis approach [23] or the methods of [7, 8] or [79],

$$\bar{p}_i(v,w,t_i)(1 - t_i s(v,w)) + \sum_{k=1}^{m+1} p_{i,k}(v,w,t_i) g_{i,k}(v,w) = 1 \tag{4.75}$$

or alternatively, when $s(v,w) = s_1(v,w)\cdots s_l(v,w)$, solve

$$\sum_{j=1}^{l} \bar{p}_{i,j}(v,w,\mathbf{t}_i)(1 - t_{ij} s_j(v,w)) + \sum_{k=1}^{m+1} p_{i,k}(v,w,\mathbf{t}_i) g_{i,k}(v,w) = 1 \tag{4.76}$$

where $\mathbf{t}_i = (t_{i1},\dots,t_{il})$. If no solutions to Equation (4.75) or (4.76) can be found, then Equation (4.1) is insolvable.

step 5. Substitute $t_i = 1/s(v,w)$ into Equation (4.75) or $t_{ij} = 1/s_j(v,w)$, $j = 1,\dots,l$ into Equation (4.76), and clear out the denominators to obtain

$$\sum_{j=1}^{m+1} \tilde{x}_{i,j}(v,w) g_{i,j}(v,w) = s^{r_i}(v,w) \tag{4.77}$$

or

$$\sum_{j=1}^{m+1} \tilde{x}_{i,j}(v,w) g_{i,j}(v,w) = s_1^{r_{i1}}(v,w) s_2^{r_{i2}}(v,w) \cdots s_l^{r_{il}}(v,w) \tag{4.78}$$

step 6. Compute the solution of Equation (4.63) or Equation (4.64) as

$$[x_{i,1} \;\; x_{i,2} \cdots x_{i,m+n}] = [\tilde{x}_{i,1} \;\; \tilde{x}_{i,2} \cdots \tilde{x}_{i,m+1}] \bar{F}_i^T(v,w) \tag{4.79}$$

step 7. Eventually, we have the solution $X(v,w)$ and $Y(v,w)$ for Equation (4.68) or (4.69) (both implying solution to Equation (4.1)) as

$$[X^T(v,w) \;\; Y^T(v,w)] = [x_{i,j}(v,w)], \quad i = 1,\dots,n, \quad j = 1,\dots,m+n \tag{4.80}$$

where $x_{i,j}$ are obtained in step 6, and $\Phi(v,w)$ as in the form of the right-hand side of Equation (4.68) or (4.69).

Several features of Algorithm 4.1 are observed. First, compared with the algorithm of [68], Algorithm 4.1 does not require any explicit calculation of zeros for some polynomials; and the essential computation in step 4 can be accomplished by Gröbner basis approach which is generally applicable for multivariable polynomials, hence Algorithm 4.1 is much easier for extending to nD ($n > 2$) cases. As a matter of fact, if we put an additional condition to Equation (4.33) for nD ($n > 2$) cases that the obtained right factor coprime matrices $\bar{D}(z_1,\dots,z_n)$, $\bar{N}(z_1,\dots,z_n)$ are as well right minor coprime, the conclusion of Equation (4.35) can be shown still true by the results of [15, 123] (see Theorem 2.4). Then almost all the remained arguments in the above for 2D systems apply to nD ($n > 2$)

situation as well. Secondly, compared with the algorithms of [10–15, 49], the computation of all the maximal order minors of $F(v,w)$ and the Gröbner basis of $\mathcal{I}$ is not required, and as remarked earlier, the restriction that $\Phi(v,w) = s^r(v,w)I$ is no longer necessary either. Thirdly, however, in contrast with the above features of Algorithm 4.1, it is observed that here we have to find the right factor coprime matrices $\bar{D}_i(v,w)$, $\bar{N}_i(v,w)$ and need to find the Gröbner basis of the polynomials $g_{i,1},\ldots,g_{i,m+1}$ and $(1-ts(v,w))$ (or $(1-t_{i,j}s(v,w)$, $j=1,\ldots,l)$, for $i=1,\ldots,n$ respectively, which do not seem to be easy tasks for computation.

Much more importantly, however, we notice that the form of the results shown in Equations (4.28) and (4.56) strongly suggest that Gröbner basis approach for modules over polynomial ring can be directly employed to solve our problem effectively. This leads to an algorithm as follows.

Algorithm 4.2

Input: $F(v,w) = [D(v,w) \;\; N(v,w)] \in \mathbf{R}[v,w]^{n\times(n+m)}$, a stable polynomial $s(v,w) \in \mathbf{R}[v,w]$ vanishing over $\mathcal{V}(\mathcal{I})$.

Output: $X(v,w)$, $Y(v,w)$ and $\Phi(v,w)$ for Equation (4.1).

step 1. Calculate a Gröbner basis $G' = \{\vec{g}'_1,\ldots,\vec{g}'_{q_0}\}$ for the module generated by $\vec{f}_1,\ldots,\vec{f}_{m+n}$ that correspond to the columns of $F(v,w)$. For this purpose, the algorithms proposed in [43, 73] can be applied (see also [105]).

For $i=1,\ldots,n$ do step 2 $\sim$ step 5. (Here we only consider the general situation when $s(v,w)$ is given as $s(v,w) = s_1(v,w)\cdots s_l(v,w)$.)

step 2. Add the n-tuples

$$\vec{h}_{ik} = [0 \;\cdots\; 0 \;\; 1-t_{ik}s_k(v,w) \;\; 0\cdots 0]^T, \qquad k=1,\ldots,l \tag{4.81}$$

$\uparrow$
the ith position

to G', then calculate a Gröbner basis $G_i = \{\vec{g}_{i1},\ldots,\vec{g}_{iq_i}\}$ for the module generated by $\{\vec{g}'_1,\ldots,\vec{g}'_{q_0},\vec{h}_{i1},\ldots,\vec{h}_{il}\}$.

step 3. By tracing the construction procedure performed in step 1 and step 2, construct $\tilde{u}_{i,j,k}(v,w,\mathbf{t}_i)$ and $\tilde{u}_{i,j,r}(v,w,\mathbf{t}_i) \in \mathbf{R}[v,w,\mathbf{t}_i]$, with $\mathbf{t}_i = (t_{i1},\ldots,t_{il})$, $k=1,\ldots,m+n$, $r=1,\ldots,l$, such that

$$\vec{g}_{ij} = \sum_{k=1}^{m+n}\tilde{u}_{i,j,k}\vec{f}_k + \sum_{r=1}^{l}\tilde{u}_{i,j,r}\vec{h}_{il}, \quad j=1,\ldots,q_i \tag{4.82}$$

Note: Denote by $\vec{e}_i$ the n-tuple having 1 as the element at the ith position and zeros at the other positions. Then, by the properties of Gröbner basis for modules, Equation (4.56) is solvable if and only if $\vec{e}_i$ can be reduced to zero with respect to G_i. According to this fact and Theorem 4.3, therefore, when Equation (4.1) is solvable, there must be a $\vec{g}_{ij}$ for certain j such that $\vec{g}_{ij} = \gamma\vec{e}_i$ with $\gamma \in \mathbf{R}$. Without loss of generality, we assume that $\gamma = 1$. If this is not true, then Equation (4.1) has no solution. This can also serve as an alternative test for the stabilizability.

step 4. Pick out from $G_i = \{\vec{g}_{i1},\ldots,\vec{g}_{iq_i}\}$ the element, say $\vec{g}_{ib}$, $b \in \{1,\ldots,q_i\}$, that satisfies

$$\vec{e}_i = \vec{g}_{ib} = \sum_{k=1}^{m+n}\tilde{u}_{i,b,k}\vec{f}_k + \sum_{r=1}^{l}\tilde{u}_{i,b,r}\vec{h}_{il} \tag{4.83}$$

then by comparing Equation (4.83) with Equation (4.56), we have

$$\bar{x}_{i,j}(v,w,\mathbf{t}_i) = \tilde{u}_{i,b,j}(v,w,\mathbf{t}_i), \qquad j=1,\ldots,m+n \tag{4.84}$$

$$\bar{x}_{i,k}(v,w,\mathbf{t}_i) = \tilde{u}_{i,b,k}(v,w,\mathbf{t}_i) \qquad k=1,\ldots,l \tag{4.85}$$

step 5. Substituting $t_{ij} = 1/s_j(v,w)$, $j=1,\ldots,l$ into Equation (4.56) and clearing out the denominators, we obtain the solution $x_{i,j}(v,w)$, $j=1,\ldots,m+n$, for Equation (4.64).

step 6. Finally, from Equation (4.64) we have a solution $X(v,w)$, $Y(v,w)$ to Equation (4.1) as

$$[X^T(v,w) \;\; Y^T(v,w)] = [x_{i,j}(v,w)], \quad i=1,\ldots,n,\; j=1,\ldots,m+n \tag{4.86}$$

and a stable $\Phi(v,w)$ in the form of the right-hand side of Equation (4.69).

By Theorem 4.3 and Algorithm 4.2, we see that Equation (4.1) can be equivalently transformed to a Bezout equation, and consequently a particular solution to the equation can be obtained by nothing else than finding the Gröbner bases for certain modules.

It is trivial to see that the general solution for Equation 4.1 can be given as follows [62, 92].

Theorem 4.4 *Let $\bar{X}$, $\bar{Y} \in \mathbf{M}(\mathbf{R}[v,w])$ be a solution of Equation* 4.1. *Then the general solution of Equation* 4.1 *is*

$$\begin{cases} X(v,w) = \bar{X}(v,w) + \bar{N}(v,w)T(v,w) \\ Y(v,w) = \bar{Y}(v,w) - \bar{D}(v,w)T(v,w) \end{cases} \tag{4.87}$$

where $\bar{D}$, $\bar{N} \in \mathbf{M}(\mathbf{R}[v,w])$ are right factor coprime matrices satisfying

$$\bar{N}(v,w)\bar{D}^{-1}(v,w) = D^{-1}(v,w)N(v,w) \tag{4.88}$$

and $T \in \mathbf{M}(\mathbf{R}[v,w])$ is arbitrary.

To be complete, the causality of the solution has yet to be considered.

Theorem 4.5 *Let $D^{-1}(v,w)N(v,w)$ be causal, i.e.,* $\det D(0,0) \neq 0$. *Then Equation* 4.1 *possesses a strictly causal solution such that* $\det X(0,0) \neq 0$ *and* $Y(0,0) = 0$ *whenever it is solvable.*

Proof. See [68]. □

4.4 Ω-Coprime MFD on Stable Rational Ring H

The algebraic structure {**G, H, I, J**} developed by Desoer et al. [27] will be applied to the 2D situation here. Let **G** be the commutative ring of 2D causal rational functions in v and w, i.e.,

$$\mathbf{G} = \{n/d \mid n, d \in \mathbf{R}[v,w],\ d(0,0) \neq 0\} \tag{4.89}$$

and let **H** be the ring of the 2D causal rational functions with no poles in an "unstable" or "undesirable" region $\Omega \subset \mathbf{C}^2$. **H** is called the Ω-stable rational ring by some authors (see [79]) and it can be shown that **H** is a subring of **G**.

The domain Ω is selected as in [79]. Namely, let Ω have the decomposition

$$\Omega = \Omega_v \times \Omega_w \tag{4.90}$$

where $\Omega_v \subset \mathbf{C}$, $\Omega_w \subset \mathbf{C}$. For the purpose of solvability over $\mathbf{R}[v,w]$, Ω_v and Ω_w are assumed to be symmetric with respect to the real axis, i.e.,

$$v \in \Omega_v \Rightarrow \bar{v} \in \Omega_v, \qquad w \in \Omega_w \Rightarrow \bar{w} \in \Omega_w \tag{4.91}$$

where $\bar{v}$ and $\bar{w}$ are the complex conjugates of v and w, respectively. It has been pointed out by Raman and Liu [79] that the usual unstable regions of interest in 2D systems satisfy Equations (4.90) and (4.91). For example, $\Omega_1 = \{|v| \leq 1\} \times \{|w| \leq 1\}$ for internal or structural stability of 2D discrete systems (see Section 3.5 or [10, 49]), $\Omega_2 = \mathbf{C}^2$ if we seek to obtain 2D FIR (Finite Impulse Response) systems. Taking into account that the causality of the transfer functions of these systems are defined just as in Equation (4.89) (see Section 3.3) and that the corresponding unstable region involves the origin, one can conclude that the defined **G** and **H** apply to these systems and contain 2D polynomials as a subring. It will become clear (in Chapter 5) that this property offers us possibility and convenience of solving asymptotic and deadbeat servo control problems in a unified way. There are also other types of 2D systems which are often encountered in practice, such as proper delay-differential systems [56, 75]. The unstable region for this class of systems is $\Omega_3 = \{\mathrm{Re}\, v \geq 0\} \times \{|w| \leq 1\}$ [56], which obviously satisfy Equations (4.90) and (4.91). But the properness of a delay-differential system, say $t(v,w)$, is usually defined with respect to v as $t(\infty, w) < \infty$, and it does not agree with the concept of "causality" considered here. Therefore, the transfer functions of proper delay-differential systems cannot be directly viewed as elements of **G** and **H**. We can, however, modify the situation to the desired one by defining the mapping

$$z = f(v) = 1/v \tag{4.92}$$

and

$$\Omega_{z_3} = f(\Omega_{v_3}) \tag{4.93}$$

Then we get a region $\Omega_3^* = \Omega_{z_3} \times \Omega_{w_3}$ which still satisfy Equations (4.90) and (4.91), and we can now interpret the properness with respect to v as the "causality" with respect to z in the sense of Equation (4.89), to which the presented approach can be applied.

Let **I** denote the set of elements of **H** which are units of **G**, i.e.,

$$\mathbf{I} = \{h \in \mathbf{H} \mid h^{-1} \in \mathbf{G}\} \tag{4.94}$$

and let **J** be the subgroup of **H** consisting of all invertible elements of **H**, i.e.,

$$\mathbf{J} = \{h \in \mathbf{H} \mid h^{-1} \in \mathbf{H}\} \tag{4.95}$$

Note that $\mathbf{J} \subset \mathbf{I} \subset \mathbf{H} \subset \mathbf{G}$. Moreover, as defined previously, $\mathbf{M}(*)$ corresponds to the set of matrices with entries in $*$ (e.g. $\mathbf{G}$, $\mathbf{H}$). A member of $\mathbf{M}(\mathbf{H})$ is said to be $\mathbf{G}$-unimodular (respectively $\mathbf{H}$-unimodular) if and only if it is square and its determinant belongs to $\mathbf{I}$ ($\mathbf{J}$).

Given the above structure, it is said that a given $P \in \mathbf{M}(\mathbf{G})$ has a right matrix fractional description (MFD) in $\{\mathbf{G}, \mathbf{H}, \mathbf{I}, \mathbf{J}\}$ if there exist D_r, $N_r \in \mathbf{M}(\mathbf{H})$ and D_r is $\mathbf{G}$-unimodular such that $P = N_r D_r^{-1}$.

In addition, let D_r, $N_r \in \mathbf{M}(\mathbf{H})$. Then $F \in \mathbf{M}(\mathbf{H})$ is a right common factor of D_r, N_r over $\mathbf{H}$ if and only if there exist Q, $R \in \mathbf{M}(\mathbf{H})$ such that $D_r = QF$, $N_r = RF$. Now two definitions will be given which may be viewed as the correspondents in $\mathbf{H}$ to the concepts of factor coprimeness and zero coprimeness in 2D polynomial ring.

Definition 4.1 *Let D_r, $N_r \in \mathbf{M}(\mathbf{H})$. Then D_r and N_r are right factor Ω-coprime if and only if every right common factor of D_r, N_r is $\mathbf{H}$-unimodular.*

Definition 4.2 *D_r and $N_r \in \mathbf{M}(\mathbf{H})$ are right zero Ω-coprime if and only if there exist U, $V \in \mathbf{M}(\mathbf{H})$ such that*

$$UD_r + VN_r = I \tag{4.96}$$

Then a right MFD $N_r D_r^{-1}$ in $\{\mathbf{G}, \mathbf{H}, \mathbf{I}, \mathbf{J}\}$ is said to be a right zero Ω-coprime MFD if and only if D_r and N_r are right zero Ω-coprime.

The dual definitions on left can be given analogously.

It has been known that a zero Ω-coprime MFD of $P \in \mathbf{M}(\mathbf{G})$ need not exist in general (see [27, 104]). The existence of such factorizations has been considered by some authors from a state-space point of view (see, e.g., [56, 58]). Now as a straightforward application of the results obtained in the previous section (or the other existing algebraic results of, e.g., [10, 49, 68]), a necessary and sufficient condition is given for the existence of such fractional description in the 2D case. First, it is observed that one can always represent an arbitrary 2D causal rational matrix $P \in \mathbf{M}(\mathbf{G})$ as $P = N_r D_r^{-1}$ where D_r, $N_r \in \mathbf{M}(\mathbf{R}[v, w]) \subset \mathbf{M}(\mathbf{H})$. Since it has been shown by Kamen [56] that when studying stable factorizations one may take D_r and N_r to be polynomial matrices without loss of generality, we will restrict our attention to this case.

Theorem 4.6 *Let D_r, $N_r \in \mathbf{M}(\mathbf{R}[v, w])$. Then D_r, N_r are right zero Ω-coprime if and only if the maximal order minors of $[D_r^T \; N_r^T]^T$ have no common zeros in Ω.*

Proof. Since D_r and N_r have not been assumed to be right factor coprime, they may share a right common factor R, i.e., $D_r = \bar{D}_r R$, $N_r = \bar{N}_r R$ for some $\bar{D}_r$, $\bar{N}_r \in \mathbf{M}(\mathbf{R}[v, w])$. If such a factor exists, it can be found by using the methods of [74] or [48] as D_r and N_r are 2D polynomial matrices. By the assumption that the maximal order minors of $[D_r^T \; N_r^T]^T = [\bar{D}_r^T \; \bar{N}_r^T]^T R$ have no common zeros in Ω and the Cauchy-Binet theorem, R must be $\mathbf{H}$-unimodular and the maximal order minors of $[\bar{D}_r^T \; \bar{N}_r^T]^T$ have no common zeros in Ω.

Then, by using the daul version of the algorithms proposed in the previous section, matrices X, Y, $\Phi \in \mathbf{M}(\mathbf{R}[v, w])$ can be constructively found such that the following equation holds if and only if the maximal order minors of $[\bar{D}_r^T \; \bar{N}_r^T]^T$ have no common zeros in Ω.

$$X\bar{D}_r + Y\bar{N}_r = \Phi \tag{4.97}$$

where $\det \Phi$ is devoid of zeros in Ω. Letting $\bar{U} = \Phi^{-1}X$, $\bar{V} = \Phi^{-1}Y \in \mathbf{M}(\mathbf{H})$, we get

$$\bar{U}\bar{D}_r + \bar{V}\bar{N}_r = I \tag{4.98}$$

Postmultiplying and premultiplying Equation (4.98) by R and R^{-1}, respectively, yield the solution $U = R^{-1}\bar{U}$, $V = R^{-1}\bar{V} \in \mathbf{M}(\mathbf{H})$ to the right zero Ω-coprime equation (4.96). The theorem has thus been proved. □

Theorem 4.6 provides a method to determine whether or not a right MFD $N_r D_r^{-1}$ of $P \in \mathbf{M}(\mathbf{G})$, with N_r and D_r being 2D polynomial matrices, is a right zero Ω-coprime one. It will be seen that representations of this form are of great importance for the stabilization of 2D systems.

It might be noted that if we take $\Omega = \mathbf{C}^2$, then the zero Ω-coprimeness will be equivalent to the zero coprimeness and the factor Ω-coprimeness equivalent to the factor coprimeness. If it is not the case, the zero coprimeness implies zero Ω-coprimeness and factor coprimeness implies factor Ω-coprimeness but not vise versa.

Furthermore, zero Ω-coprimeness need not imply factor coprimeness since two zero Ω-coprime matrices may have a $\mathbf{H}$-unimodular common factor. Conversely, factor coprimeness does not imply zero Ω-coprimeness either as two factor coprime matrices may not be zero Ω-coprime if they do not satisfy the condition of Theorem 4.6.

Based on the right and left zero Ω-coprime MFD's of 2D causal rational matrices, a 2D doubly zero Ω-coprime relation can be also obtained.

Theorem 4.7 *Suppose $P \in \mathbf{M}(\mathbf{G})$, and let $D_l^{-1}N_l$, $N_rD_r^{-1}$ be any left zero Ω-coprime MFD and right zero Ω-coprime MFD of P respectively. Then there exist U, V, $\tilde{U}$, $\tilde{V} \in$ $\mathbf{M}(\mathbf{H})$ such that*

$$\begin{bmatrix} U & V \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -\tilde{V} \\ N_r & \tilde{U} \end{bmatrix} = I \tag{4.99}$$

Proof. The proof is similar to the 1D case (see, e.g., [84, 103]). □

4.5 Parametrization of All 2D Stabilizing Compensators

Consider an MIMO 2D causal LSI system as shown in Figure 4.1, where $P \in \mathbf{M}(\mathbf{G})$ is the plant to be controlled, $C \in \mathbf{M}(\mathbf{G})$ is the controller to be found. Let $N_pD_p^{-1}$ be a right zero Ω-coprime MFD of P with N_p, $D_p \in \mathbf{M}(\mathbf{H})$, and $D_c^{-1}[N_{c1} \;\; N_{c2}]$ be a left zero Ω-coprime MFD of C with D_c, N_{c1}, $N_{c2} \in \mathbf{M}(\mathbf{H})$.

Because

$$e_2 = D_pz_2 = y_1 + u_2 \Rightarrow y_1 - D_pz_2 = -u_2$$

and

$$z_1 = D_py_1 = N_{c1}u_1 - N_{c2}(u_3 + y_2) = N_{c1}u_1 - N_{c2}u_3 - N_{c2}N_pz_2$$

$$\Rightarrow D_cy_1 + N_{c2}N_pz_2 = N_{c1}u_1 - N_{c2}u_3,$$

the following relation can be obtained [103]:

$$\begin{bmatrix} I & -D_p \\ D_c & N_{c2}N_p \end{bmatrix} \begin{bmatrix} y_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ N_{c1} & 0 & -N_{c2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.100}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N_p \end{bmatrix} \begin{bmatrix} y_1 \\ z_2 \end{bmatrix} \tag{4.101}$$

Let

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Then from Equations (4.100) and (4.101), we have

$$y = H_{yu}u \tag{4.102}$$

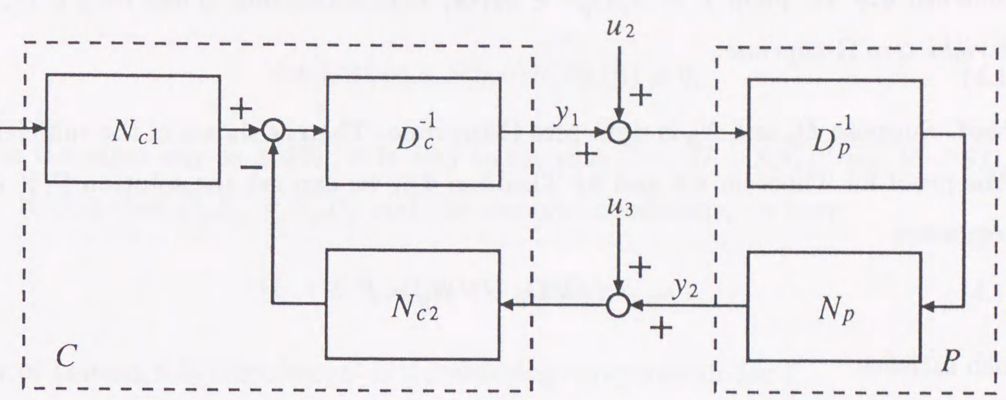


Figure 4.1: 2D Feedback Control System

where

$$H_{yu} = \begin{bmatrix} I & 0 \\ 0 & N_p \end{bmatrix} \begin{bmatrix} I & -D_p \\ D_c & N_{c2}N_p \end{bmatrix}^{-1} \begin{bmatrix} 0 & -I & 0 \\ N_{c1} & 0 & -N_{c2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.103}$$

It is trivial to see

$$\begin{bmatrix} I & -D_p \\ D_c & N_{c2}N_p \end{bmatrix}^{-1} = \begin{bmatrix} I - D_p\Delta^{-1}D_c & D_p\Delta^{-1} \\ -\Delta^{-1}D_c & \Delta^{-1} \end{bmatrix} \tag{4.104}$$

where

$$\Delta = N_{c2}N_p + D_cD_p. \tag{4.105}$$

Hence, we have that

$$H_{yu} = \begin{bmatrix} D_p\Delta^{-1}N_{c1} & -I + D_p\Delta^{-1}D_c & -D_p\Delta^{-1}N_{c2} \\ N_p\Delta^{-1}N_{c1} & N_p\Delta^{-1}D_c & -N_p\Delta^{-1}N_{c2} \end{bmatrix} \tag{4.106}$$

Definition 4.3 *If $\det\Delta \neq 0$ and $H_{yu} \in \mathbf{M}(\mathbf{H})$, we say that the 2D feedback system of Figure 4.1 is Ω-stable. Further, if there exists $C \in \mathbf{M}(\mathbf{G})$ such that $H_{yu} \in \mathbf{M}(\mathbf{H})$, we say that P is Ω-stabilizable and that C is a Ω-stabilizing compensator for P.*

Lemma 4.3 *The 2D feedback system of Figure 4.1 is Ω-stable if and only if Δ is $\mathbf{H}$-unimodular.*

Proof. The sufficiency is obvious. The necessity can be shown in the same way as [103, 104] under the conditions that D_p, N_p are right zero Ω-coprime and D_c, $[N_{c1} \;\; N_{c2}]$ are left zero Ω-coprime. □

Theorem 4.8 2D *plant* $P = N_pD_p^{-1} \in \mathbf{M}(\mathbf{G})$ *is* Ω*-stabilizable if and only if* D_p *and* N_p *are right zero* Ω*-coprime.*

Proof. Suppose D_p and N_p is right zero Ω-coprime. Then as shown in the sufficiency part of the proof for Theorem 4.6 and by Theorem 4.5, we can get the solution $U, V \in \mathbf{M}(\mathbf{H})$ to equation

$$UD_p + VN_p = I \tag{4.107}$$

which satisfies

$$\det U(0,0) \neq 0. \tag{4.108}$$

Hence, $C = U^{-1}[N_{c1}\ V] \in \mathbf{M}(\mathbf{G})$ for any $N_{c1} \in \mathbf{M}(\mathbf{H})$. By the fact that I is $\mathbf{H}$-unimodular and in view of Lemma 4.3, the sufficiency is established.

Conversely, if $C = X^{-1}Y = X^{-1}[Y_1\ \ Y_2] \in \mathbf{M}(\mathbf{G})$ is a Ω-stabilizing compensator of P, then by Lemma 4.3 the relation

$$XD_p + Y_2N_p = \Phi \tag{4.109}$$

holds where X, Y_2, $\Phi \in \mathbf{M}(\mathbf{H})$ and Φ is $\mathbf{H}$-unimodular. Premultiplying (4.109) by Φ^{-1} gives

$$(\Phi^{-1}X)D_p + (\Phi^{-1}Y_2)N_p = I \tag{4.110}$$

which shows that D_p and N_p is right zero Ω-coprime. □

The following theorem gives the parametrization of all nD Ω-stabilizing compensators.

Theorem 4.9 *Suppose that* $N_pD_p^{-1}$ *and* $\tilde{D}_p^{-1}\tilde{N}_p$ *are any right and left zero* Ω*-coprime MFD for a given plant* $P \in \mathbf{M}(\mathbf{G})$*, respectively. Let* $U, V \in \mathbf{M}(\mathbf{H})$ *such that* $UD_p + VN_p = I$*. Then the set of all* Ω*-stabilizing compensators of* P *is given by*

$$C \in \{(U + S\tilde{N}_p)^{-1}[Q\ \ V - S\tilde{D}_p] \mid Q, S \in \mathbf{M}(\mathbf{H}),\ \det(U + S\tilde{N}_p) \in \mathbf{I}\} \tag{4.111}$$

and the set of all possible Ω*-stable transfer matrices from* u *to* y *is in the form*

$$\begin{bmatrix} D_pQ & D_p(U + S\tilde{N}_p) - I & -D_p(V - S\tilde{D}_p) \\ N_pQ & N_p(U + S\tilde{N}_p) & -N_p(V - S\tilde{D}_p) \end{bmatrix} \tag{4.112}$$

Proof. Let $S \in \mathbf{M}(\mathbf{H})$ such that $\det(U + S\tilde{N}_p) \in \mathbf{I}$, i.e.,

$$\det(U(0,0) + S(0,0)\tilde{N}_p(0,0)) \neq 0. \tag{4.113}$$

Then, in the same way as in [49], it is easy to see that $C = (U + S\tilde{N}_p)^{-1}[Q\ \ V - S\tilde{D}_p] \in \mathbf{M}(\mathbf{G})$. Noting that $\tilde{D}_pN_p = \tilde{N}_pD_p$ and the assumed conditions, we have

$$(U + S\tilde{N}_p)D_p + (V - S\tilde{D}_p)N_p = I. \tag{4.114}$$

In view of Lemma 4.3, therefore, C is a stabilizing compensator for P.

Conversely, suppose that C

$$C = \bar{X}^{-1}\bar{Y} \triangleq \bar{X}^{-1}[\bar{Y}_1\ \ \bar{Y}_2] \tag{4.115}$$

stabilizes P. Then, by Lemma 4.3,

$$\bar{X}D_p + \bar{Y}_2N_p = \Phi \tag{4.116}$$

where $\Phi \in \mathbf{M}(\mathbf{H})$ is $\mathbf{H}$-unimodular. Therefore, it follows directly that

$$XD_p + Y_2N_p = I \tag{4.117}$$

where $X = \Phi^{-1}\bar{X}$ and $Y_2 = \Phi^{-1}\bar{Y}_2$.

According to Theorem 4.7, there exist U, V, $\tilde{U}$, $\tilde{V} \in \mathbf{M}(\mathbf{H})$ such that

$$\begin{bmatrix} D_p & \tilde{V} \\ N_p & -\tilde{U} \end{bmatrix}^{-1} = \begin{bmatrix} U & V \\ \tilde{N}_p & -\tilde{D}_p \end{bmatrix}. \tag{4.118}$$

Then, we have

$$[X\ \ Y_2]\begin{bmatrix} D_p & \tilde{V} \\ N_p & -\tilde{U} \end{bmatrix} = [I\ \ S] \tag{4.119}$$

where $S = X\tilde{V} - Y_2\tilde{U}$. This in fact implies that

$$\begin{aligned}[X\ \ Y_2] &= [I\ \ S]\begin{bmatrix} U & V \\ \tilde{N}_p & -\tilde{D}_p \end{bmatrix} \\ &= [U + S\tilde{N}_p\ \ V - S\tilde{D}_p]. \end{aligned} \tag{4.120}$$

Now it is ready to see

$$C = \bar{X}^{-1}\bar{Y}$$

$$= (\Phi^{-1}\bar{X})^{-1}[\Phi^{-1}\bar{Y}_1 \;\; \Phi^{-1}\bar{Y}_2]$$

$$= X^{-1}[Q \;\; Y_2]$$

$$= (U + S\tilde{N}_p)^{-1}[Q \;\; V - S\tilde{D}_p] \tag{4.121}$$

where $Q = \Phi^{-1}\bar{Y}_1 \in \mathbf{M(H)}$, and the relation (4.112) follows by routine computation. □

4.6 Example

Consider the 2D system

$$P(v,w) = \begin{bmatrix} -\dfrac{w-3v}{2v-5} & \dfrac{2v-5}{3(2v-1)} \\ \dfrac{2v-1}{8w+6v-15} & \dfrac{w^2}{2v-1} \end{bmatrix} \tag{4.122}$$

and let $\Omega = \bar{U}^2$. We can represent $P(v,w)$ by the left factor coprime MFD

$$P(v,w) = D^{-1}(v,w)N(v,w) \tag{4.123}$$

where

$$D(v,w) = \begin{bmatrix} 3(2v-1)(2v-5)/16 & 0 \\ -3w^2(2w-3)(2v-5)/2 & 2(8w+6v-15) \end{bmatrix},$$

$$N(v,w) = \begin{bmatrix} -3(2v-1)(w-3v)/16 & (2v-5)^2/16 \\ (6w^4-18w^3v-9w^3+27w^2v+8v-4)/2 & -w^2((2w-3)(2v-5)-8w)/2 \end{bmatrix}.$$

First, by using the results given in Section 4.2, we investigate the stabilizability of $P(v,w)$ and construct a stable polynomial $s(v,w)$ vanishing on $\mathcal{V}(\mathcal{I})$. Applying the result of Corollary 2.1 to the above $D(v,w)$ and $N(v,w)$, the solutions to Equations (4.5) and (4.6) can be obtained with the following $\Phi_1(v)$ and $\Phi_2(w)$. (For brevity, the results for $X_1(v,w)$, $Y_1(v,w)$, $X_2(v,w)$ and $Y_2(v,w)$ are omitted here.)

$$\Phi_1(v) = \begin{bmatrix} 3(2v-1)^2(2v-5) & 0 \\ 0 & 6(2v-1)^2(2v-5) \end{bmatrix},$$

$$\Phi_2(w) = \begin{bmatrix} 96(2w-3)(2w-15)w & 0 \\ 0 & 64(2w-3)(2w-15)w \end{bmatrix}.$$

Then, $\Phi_1(v)$ and $\Phi_2(w)$ can be decomposed as in Equations (4.7) and (4.8) with

$$\Phi_{1u}(v) = \begin{bmatrix} 3(2v-1)^2 & 0 \\ 0 & 6(2v-1)^2 \end{bmatrix},$$

$$\Phi_{1s}(v) = \begin{bmatrix} (2v-5) & 0 \\ 0 & (2v-5) \end{bmatrix},$$

$$\Phi_{2u}(w) = \begin{bmatrix} 96w & 0 \\ 0 & 64w \end{bmatrix},$$

$$\Phi_{2s}(w) = \begin{bmatrix} (2w-3)(2w-15) & 0 \\ 0 & (2w-3)(2w-15) \end{bmatrix}.$$

It is easy to see

$$\Gamma\{\det \Phi_{1u}(v)\} \cap \Gamma\{\det \Phi_{2u}(w)\} = \{(1/2, 0)\} \tag{4.124}$$

By checking

$$\text{rank}\,[D(1/2,0) \;\; N(1/2,0)] = 2, \tag{4.125}$$

we conclud that $(1/2, 0)$ is not a common zero of all the 2×2 minors of $[D(v,w) \;\; N(v,w)]$, and according to Theorem 4.1 $P(v,w)$ is stabilizable.

On the other hand, the stable polynomial $s(v,w)$ can be constructed by using all the mutually coprime factors in $\det \Phi_{1s}(v)$ and $\det \Phi_{2s}(w)$ as

$$s(v,w) = (2v-5)(2w-3)(2w-15). \tag{4.126}$$

Now, following Algorithm 4.2, we can solve Equation (4.1) by using the Gröbner approach for modules. Let

$$F(v,w) = [D(v,w) \;\; N(v,w)], \tag{4.127}$$

so we have

$$\vec{f}_1 = [3(2v-1)(2v-5)/16 \;\; -3w^2(2w-3)(2v-5)/2]^T,$$

$$\vec{f}_2 = [0 \;\; 2(8w+6v-15)]^T,$$

$$\vec{f}_3 = [-3(2v-1)(w-3v)/16 \;\; (6w^4-18w^3v-9w^3+27w^2v+8v-4)/2]^T,$$

$$\vec{f}_4 = [(2v-5)^2/16 \;\; -w^2((2w-3)(2v-5)-8w)/2]^T.$$

In addition, let

$$\vec{h}_1 = \begin{bmatrix} 1 - t_1 s(v,w) \\ 0 \end{bmatrix}, \quad \vec{h}_2 = \begin{bmatrix} 0 \\ 1 - t_2 s(v,w) \end{bmatrix}.$$

By the method of, for example, [43], we can find the Gröbner basas G_1 and G_2 for the modules generated by $\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4, \vec{h}_1\}$ and $\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4, \vec{h}_2\}$ respectively as

$$G_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}, \tag{4.128}$$

$$G_2 = \{ \begin{bmatrix} w - 15/2 \\ 0 \end{bmatrix}, \begin{bmatrix} v - 5/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \tag{4.129}$$

and obtain the results

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sum_{k=1}^{4} \tilde{x}_{1,k}(v, w, t_1)\vec{f}_k(v, w) + \bar{x}_1(v, w, t_1)\vec{h}_1(v, w, t_1), \tag{4.130}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sum_{k=1}^{4} \tilde{x}_{2,k}(v, w, t_2)\vec{f}_k(v, w) + \bar{x}_2(v, w, t_2)\vec{h}_2(v, w, t_2) \tag{4.131}$$

where

$$\tilde{x}_{1,1} = (1 - 12(2w - 15)(2v - 5)t_1)(27w^3 - 81w^2v + 32w + 24v - 108)(2w - 3)t_1/12,$$

$$\tilde{x}_{1,2} = (1 - 12(2w - 15)(2v - 5)t_1)(8w - 6v + 15)(2w - 3)w^2t_1/4,$$

$$\tilde{x}_{1,3} = 9(1 - 12(2w - 15)(2v - 5)t_1)(2w - 3)(2v - 5)w^2t_1/4,$$

$$\tilde{x}_{1,4} = -(1 - 12(2w - 15)(2v - 5)t_1)(8w + 6v - 15)(2w - 3)t_1,$$

$$\bar{x}_1 = 1 - 12(2w - 3)(2v - 5)t_1$$

$$\tilde{x}_{2,1} = 9(1 - 12(2w - 15)(2v - 5)t_2)(w - 3v)(2v - 1)t_2/32,$$

$$\tilde{x}_{2,2} = (1 - 12(2w - 15)(2v - 5)t_2)(8w - 6v - 9)(2v - 5)t_2/32,$$

$$\tilde{x}_{2,3} = 9(1 - 12(2w - 15)(2v - 5)t_2)(2v - 1)(2v - 5)t_2/32,$$

$$\tilde{x}_{2,4} = 0,$$

$$\bar{x}_2 = 1 - 12(2w - 3)(2v - 5)t_2.$$

By substituting $t_i = 1/s,\ i = 1, 2$ into Equations (4.92) and (4.93) and clearing out the denominators, we can get the solution to Equation (4.1), i.e.,

$$D(v, w)X(v, w) + N(v, w)Y(v, w) = \Phi(v, w)$$

with

$$X(v, w) = \begin{bmatrix} 27w^3 - 81w^2v + 32w + 24v - 108 & 9(w - 3v)(2v - 1) \\ 3(8w - 6v + 15)w^2 & (8w - 6v - 9)(2v - 5) \end{bmatrix}$$

$$Y(v, w) = \begin{bmatrix} 27w^2(2v - 5) & 9(2v - 1)(2v - 5) \\ -12(8w + 6v - 15) & 0 \end{bmatrix}$$

$$\Phi(v, w) = \begin{bmatrix} 12(2w - 3)(2v - 5) & 0 \\ 0 & 32(2w - 3)^2(2v - 5) \end{bmatrix}$$

4.7 Summary

Some alternative methods have been proposed for testing the 2D output feedback stabilizability and constructing a 2D stable closed-loop polynomial. By these methods, the problems are reduced to the 1D case and can be solved by using 1D algorithms or Gröbner basis approach. The idea adopted here can be applied to establish similar procedures for nD ($n > 2$) cases under certain conditions (which will be discussed further in Chapter 8).

Moreover, some generalizations for the "Rabinowitsch trick" have been obtained in the senses of Lemmas 4.1, 4.2 and Theorem 4.3. These results eventually lead to two new constructive algorithms, Algorithm 4.1 and Algorithm 4.2, for solving Equation (4.1). Although Algorithm 4.1 plays an important role in the proofs of the above results and has some advantages over the existing procedures, it may also possess some computational problems as discussed earlier. Complementally, however, Algorithm 4.2 suggests that one can solve an equivalent Bezout equation by direct application of the Gröbner basis approach for modules in order to obtain the solution to Equation (4.1). In consequence, Equation (4.1) can be effectively solved with neither computation of any minors or zeros nor estimation of any degrees, but yet a general form for $\Phi(v, w)$ as shown in [68] can be achieved.

Recently, Shankar and Sule [95] have developed a general theory of feedback stabilization for systems described by transfer functions over a general integral domain which extends the well-known coprime factorization approach. By using this theory, then, they clarified necessary and sufficient condition for the solvability of nD stabilization problem in terms of affine varieties. It seems, however, that the task remains of finding some constructive and effective algorithms to test the conditions and to construct the stabilizing compensators. Since the algorithms proposed here have good potentiality for extension to nD ($n > 2$) cases, we believe that it would be easy to generalize the algorithms by overcoming several minor technical problems.

Chapter 5

2D Skew Ω-Primeness and Asymptotic and Deadbeat Servo Problems for 2D Systems

5.1 Introduction

It is well known that the concept of skew primeness, introduced by Wolovich [108] when studying the bilateral (1D) polynomial matrix equation $DU + VN = I(\Phi)$, where D, N and Φ are given and U, V need to be found, is essential to resolving various 1D output regulation and tracking problems (see, e.g., [63, 109]). In the context of 2D linear systems, bilateral equations in a 2D polynomial ring have been naturally encountered as well (see [53, 112]). The methods for 1D case, however, cannot be directly applied since 2D polynomial ring is non-Euclidean. The problem remained unsolved until the approach by Šebek [94] appeared recently.

Šebek [94], first, generalized the concept of skew primeness to the 2D (respectively nD) polynomial situation in two versions, namely, factor skew primeness and zero skew primeness. Then the bilateral 2D (nD) polynomial matrix equation

$$D(v,w)U(v,w) + V(v,w)N(v,w) = \Phi(v,w) \tag{5.1}$$

was solved by finding first the unique minimum degree solution in the ring of polynomials in one indeterminate having polynomial fractions in the other indeterminate(s) as coefficients, and then checking whether the obtained solution belongs to the original 2D (nD) polynomial ring. Needless to say, the case where $\Phi = I$, i.e., the zero skew prime equation

$$D(v,w)U(v,w) + V(v,w)N(v,w) = I \tag{5.2}$$

is included, which is of particular interest in our study because of its relevance in deadbeat control problems of 2D systems. To apply the above method, however, the following assumptions are required:

1. D is a full rank square matrix, i.e., D is nonsingular.
2. The invariant factors (in $\mathbf{R}[v,w]$) of D and N are mutually factor coprime or have common factors from $\mathbf{R}[v]$.
3. D is monic in w.

The first assumption is standard in 2D control problems. It has been shown that the third assumption can always be satisfied by appropriate substitution if the first assumption is true (see [92]). However, the condition of Assumption 2 is somewhat restrictive since the example below shows that we can find D and N which do not satisfy this assumption but such that Equation (5.2) admits a solution in $\mathbf{R}[v,w]$. Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & vw \end{bmatrix}, \qquad N = \begin{bmatrix} vw & 0 \\ 0 & 1 \end{bmatrix} \tag{5.3}$$

Then it is trivial to verify that

$$U = \begin{bmatrix} 1-vw & v \\ v^2w & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -v \\ -v^2w & 1-vw \end{bmatrix} \tag{5.4}$$

satisfy Equation (5.2) and clearly $\det D = \det N = vw$ violates Assumption 2. In fact, Assumption 2 is not necessary for the equation itself to be solvable. Although Šebek [94] has given, as a generalization of the result of [108], a less restrictive condition for the zero skew primeness (see Theorem 2.2 of [94]), no constructive verifying procedure was proposed. One of the main objectives of this chapter will be to present a constructive procedure to verify the zero skew primeness of two given 2D polynomial matrices under this less restrictive condition.

Moreover, in 2D system control theory, besides deadbeat regulation and tracking problems, asymptotic ones are often of interest as well (see [90, 112]). If it is the case, then the bilateral 2D polynomial matrix equation (5.1) with Φ being a suitable Ω-stable polynomial

matrix ($\Phi = \phi I$ and $\phi \in \mathbf{R}[v,w]$ and $\phi \neq 0$ in Ω) rather than a general one should be paid special attention [112]. While the problem of constructing a Ω-stable polynomial matrix Φ and finding a solution U and V to the unilateral 2D polynomial matrix equation $DU + NV = \Phi$ (or $UD + VN = \Phi$) has already been solved (see Chapter 4 and also, e.g., [11, 49, 68]), the analogous problem for the bilateral equation with Ω-stable Φ has yet to be solved. Another objective of this chapter is to find necessary and sufficient conditions and a constructive solution procedure to the above bilateral equation.

For the sake of conceptual and methodological brevity, we are going to attack the two problems discussed above in a unified way. Namely, instead of the ring of 2D polynomials, the problems will be considered over the more general ring $\mathbf{H}$ of causal Ω-stable 2D rational functions which contains the 2D polynomial ring as a subring (see Section 4.4). As a natural consequence of this approach, the zero skew prime equation may be viewed as a special case of a zero skew Ω-prime equation, which will be defined in Section 5.2 and solved in Section 5.3.

Finally, by using the results obtained for the above problems, a necessary and sufficient condition and solutions to the bilateral equation (5.1) where Φ is a general 2D polynomial matrix are given in Section 5.4.

Obviously, the basic idea employed here is borrowed from [108], so that the results obtained can also be viewed as 2D extensions of those developed by Wolovich [108] for 1D systems. It is worth noting, however, that the extension is far from trivial.

5.2 2D Skew Ω-primeness

This section gives the definition of 2D skew Ω-primeness and a necessary and sufficient condition for the 2D skew Ω-primeness.

Definition 5.1 *Consider matrices D and N in $\mathbf{M}(\mathbf{H})$. D and N will be called (externally zero) skew Ω-prime if and only if there exists a pair of 2D rational matrices U and $V \in \mathbf{M}(\mathbf{H})$ such that*

$$DU + VN = I \tag{5.5}$$

Let d and n be common denominators for the entries of D and N, respectively. Since D and $N \in \mathbf{M}(\mathbf{H})$, d and n must be in $\mathbf{J}$. Then if Equation (5.5) holds, also $D'U' + V'N' = I$, where $D' = Dd$, $N' = nN$, $U' = d^{-1}U$ and $V' = Vn^{-1}$, and clearly $D', N' \in \mathbf{M}(\mathbf{R}[v,w]) \subset$

$\mathbf{M(H)}$ and U', $V' \in \mathbf{M(H)}$. Thus when studying the skew Ω-primeness of D, N in $\mathbf{M(H)}$, one may take D and N to be polynomial matrices without loss of generality. In what follows, therefore, it is assumed that D and N are in $\mathbf{M}(\mathbf{R}[v,w])$. If we further assume that D is square and $\det D \neq 0$ (nonsingular), we can give the following theorem in a similar manner as [108], which relates the skew Ω-primeness with the zero Ω-coprimeness. As mentioned previously, the noted assumption is the usual case for 2D control problems.

Theorem 5.1 *Consider D and $N \in \mathbf{M}(\mathbf{R}[v,w])$ where D is nonsingular. D and N are (externally zero) skew Ω-prime if and only if there exists a pair of matrices in $\mathbf{M(H)}$, $\bar{N}$ and $\bar{D}$, such that*

$$ND = \bar{D}\bar{N} \tag{5.6}$$

with D and $\bar{N}$ right zero Ω-coprime and N and $\bar{D}$ left zero Ω-coprime.

Proof. Sufficiency: If D and $\bar{N}$ are right zero Ω-coprime and N and $\bar{D}$ are left zero Ω-coprime, and Equation (5.6) holds, then in view of Theorem 4.7 there exist four matrices X_1, X_2, X_3, $X_4 \in \mathbf{M(H)}$ such that the doubly zero Ω-coprime relation

$$\begin{bmatrix} X_1 & X_2 \\ -\bar{D} & N \end{bmatrix} \begin{bmatrix} \bar{N} & -X_3 \\ D & X_4 \end{bmatrix} = I \tag{5.7}$$

holds. The identity in Equation (5.7) directly gives

$$\begin{bmatrix} \bar{N} & -X_3 \\ D & X_4 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ -\bar{D} & N \end{bmatrix} = I \tag{5.8}$$

Therefore, we obtain the following relation which solves Equation (5.5).

$$DX_2 + X_4 N = I$$

Necessity: If D and N are externally zero skew Ω-prime, so that Equation (5.5) holds, then D and V must be left zero Ω-coprime (and U and N are right zero Ω-coprime). For the left MFD $D^{-1}V$, one can always find the representation

$$D^{-1}V = D"^{-1}V" = \bar{V}\bar{D}^{-1} \tag{5.9}$$

where $D"$, $V"$, $\bar{V}$ and $\bar{D} \in \mathbf{M}(\mathbf{R}[v,w])$ and $D"$, $V"$ are left factor coprime and $\bar{V}$, $\bar{D}$ are right factor coprime. In fact, since $V \in \mathbf{M(H)}$, $D \in \mathbf{M}(\mathbf{R}[v,w])$, letting q be the common denominator of the entries of V, we can take $D' = Dq$, $V' = Vq \in \mathbf{M}(\mathbf{R}[v,w])$

and clearly $D^{-1}V = D'^{-1}V'$. Then from $D'(q^{-1}U) + V'(q^{-1}N) = I$ and $q \in \mathbf{J}$, D' and V' are also left zero Ω-coprime. By extracting the greatest left common factor R of D' and V', $D" = R^{-1}D'$ and $V" = R^{-1}V'$ which are left factor coprime can be obtained. In the same way as in the proof of Theorem 4.6, it is easy to show $D"$ and $V"$ are left zero Ω-coprime as well.

Now, denote by $\mathcal{I}$ and $\bar{\mathcal{I}}$, respectively, the ideal generated by the all maximal order minors of the matrix $[D" \;\; V"]$ and the matrix $[\bar{D}^T \;\; \bar{V}^T]^T$. Then by Theorem 2.4, the algebraic variety $\mathcal{V}(\mathcal{I})$ is identical with $\mathcal{V}(\bar{\mathcal{I}})$. Therefore, if $D"$ and $V"$ are left zero Ω-coprime, i.e., $\mathcal{V}(\mathcal{I}) \cap \Omega = \emptyset$, $\bar{D}$ and $\bar{V}$ must be right zero Ω-coprime since $\mathcal{V}(\bar{\mathcal{I}}) \cap \Omega = \mathcal{V}(\mathcal{I}) \cap \Omega = \emptyset$.

Then we can find X and Y such that

$$X\bar{V} + Y\bar{D} = I \tag{5.10}$$

with X, $Y \in \mathbf{M(H)}$. Now we have

$$\begin{bmatrix} D & V \\ -X & Y \end{bmatrix} \begin{bmatrix} U & -\bar{V} \\ N & \bar{D} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -XU + YN & I \end{bmatrix} \tag{5.11}$$

Premultiplying Equation (5.11) by

$$\begin{bmatrix} I & 0 \\ XU - YN & I \end{bmatrix} \tag{5.12}$$

yields

$$\begin{bmatrix} D & V \\ -\bar{N} & \bar{U} \end{bmatrix} \begin{bmatrix} U & -\bar{V} \\ N & \bar{D} \end{bmatrix} = I \tag{5.13}$$

where $\bar{N} = X - (XU - YN)D$, $\bar{U} = Y + (XU - YN)V$.

The identity in Equation (5.13) obviously implies

$$\begin{bmatrix} U & -\bar{V} \\ N & \bar{D} \end{bmatrix} \begin{bmatrix} D & V \\ -\bar{N} & \bar{U} \end{bmatrix} = I \tag{5.14}$$

Therefore, we have

$$ND = \bar{D}\bar{N}$$

with D and $\bar{N}$ right zero Ω-coprime and N and $\bar{D}$ left zero Ω-coprime. The necessity has thus been established, whence the theorem follows. □

5.3 Solution of 2D Skew Ω-prime Equation

In order to determine whether or not two given matrices in $\mathbf{M}(\mathbf{H})$, such as N and D, are externally skew Ω-prime, a constructive procedure is desired. In view of Theorem 5.1, if $\bar{N}$ and $\bar{D}$ in Equation (5.6) can be found, they will directly imply a solution to Equation (5.5). Now, a procedure will be given for the determination of $\bar{N}$ and $\bar{D}$ under the assumption that N and $D \in \mathbf{M}(\mathbf{R}[v,w])$ and D is square and nonsingular.

It is first noted if D is $p \times p$ and nonsingular, its inverse D^{-1} can be represented as the ratio of its adjoint D^+, and its determinant $\det D = \Delta$; i.e.,

$$D^{-1} = D^+/\Delta \tag{5.15}$$

which implies that $DD^+/\Delta = I$, or that

$$[D^+]^{-1} = D/\Delta. \tag{5.16}$$

Furthermore, $ND/\Delta = N(D^+)^{-1}$ can always be represented as

$$N \cdot D/\Delta = N \cdot [D^+]^{-1} = Q^{-1}R \tag{5.17}$$

with Q and R left factor coprime. Let $\bar{G}$ be any greatest common right factor of N and D^+, and $g(v,w) = \det \bar{G}$. Since Q and R are left factor coprime, it follows that

$$\det Q \ g(v,w) = \det D^+ = \Delta^{p-1} \tag{5.18}$$

Then multiplying both sides of Equation (5.17) by Δ yields

$$ND = \Delta Q^{-1} R = \bar{Q}R \tag{5.19}$$

with $\bar{Q} = \Delta Q^{-1}$.

Since Q and R are left factor coprime and ND is a 2D polynomial matrix, then by Corollary 2.3 $\bar{Q} = \Delta Q^{-1}$ must also be a 2D polynomial matrix.

Moreover, since $\det \bar{Q} = g(v,w)\Delta$, according to the general factorization theorem (Theorem 2.5) a right factor $\hat{G}$ of $\bar{Q}$ can be found such that $\bar{Q} = \hat{Q}\hat{G}$ and $\det \hat{G} = g$ and $\det \hat{Q} = \Delta$. Then we write Equation (5.19) as

$$ND = \bar{Q}R = \hat{Q}\hat{G}R = \hat{Q}\hat{R} \tag{5.20}$$

with $\hat{R} = \hat{G}R$.

In light of these discussion, we can now establish the following result.

Theorem 5.2 *Consider any pair of 2D polynomial matrices N and D with D nonsingular. If $ND = \hat{Q}\hat{R}$, as in Equation (5.20), with $\det \hat{Q} = \det D = \Delta$, then N and D are external skew Ω-prime if and only if $\hat{Q}$ and N are left zero Ω-coprime or D and $\hat{R}$ are right zero Ω-coprime.*

Proof. Consider the most general situation where $\hat{Q}$, N possess left factors, and D, $\hat{R}$ possess right factors. Then the following relations hold in the ring of 2D polynomials.

$$\hat{Q} = T_l Q_0, \qquad N = T_l N_0 \tag{5.21}$$

$$\hat{R} = R_0 T_r, \qquad D = D_0 T_r \tag{5.22}$$

and obviously,

$$Q_0^{-1} N_0 = R_0 D_0^{-1} \tag{5.23}$$

where T_l is the greatest left common factor of $\hat{Q}$ and N; T_r is the greatest right common factor of D and $\hat{R}$; Q_0, $N_0 \in \mathbf{M}(\mathbf{R}[v,w])$ are left factor coprime; and R_0, $D_0 \in \mathbf{M}(\mathbf{R}[v,w])$ are right factor coprime.

Since $\hat{Q}$ and N are left zero Ω-coprime by assumption, in view of the proof of Theorem 4.6, we have that Q_0 and N_0 are left zero Ω-coprime and T_l is H-unimodular, i.e., $\det T_l \in \mathbf{J}$.

Then by using the result of Theorem 2.4 as in the proof of Theorem 5.1, one can conclude that if Q_0 and N_0 are left zero Ω-coprime, D_0 and R_0 must be right zero Ω-coprime.

Moreover, since the MFDs in Equation (5.23) are factor coprime, by Corollary 2.2, we conclude that

$$\det Q_0 = \det D_0. \tag{5.24}$$

From assumption, we have

$$\det \hat{Q} = \det Q_0 \cdot \det T_l = \det D = \det D_0 \cdot \det T_r. \tag{5.25}$$

Therefore,

$$\det T_r = \det T_l. \tag{5.26}$$

Since $\det T_l \in \mathbf{J}$, $\det T_r$ must also be in $\mathbf{J}$, or equivalently T_r is $\mathbf{H}$-unimodular. Now it is clear that D and $\hat{R}$ are right zero Ω-coprime.

As a result, if $\hat{Q}$ and N are left zero Ω-coprime (or, similarly, if D and $\hat{R}$ are right zero Ω-coprime), D and $\hat{R}$ must be right zero Ω-coprime ($\hat{Q}$ and N must be left zero Ω-coprime). Then in view of Theorem 5.1, the proof is complete. □

Since the zero Ω-coprime equation can be constructively solved by using the methods proposed in Chapter 4 or those of [49, 68], and in view of Theorem 5.2, the procedure to solve the skew Ω-prime equation (5.5) can now be stated as follows.

Solving Procedure:

Step 1. For given matrices $D,\ N \in \mathbf{M}(\mathbf{R}[v,w])$ where D is nonsingular, find $Q^{-1}R$ in Equation (5.17) by using the methods of [48, 74] or [67]. Then calculate $\bar{Q} = \Delta Q^{-1}$ in Equation (5.19).

Step 2. Find $\hat{Q}$ in Equation (5.20) by the general factorizing method as [74] or [48] such that $\det \hat{Q} = \Delta$, and calculate $\hat{R} = \hat{G}R$.

Step 3. By using the method of Section 4.2, the solvability of the following zero Ω-coprime equation can be verified:

$$U_1 D + V_1 \hat{R} = I \tag{5.27}$$

where $U_1,\ V_1 \in \mathbf{M}(\mathbf{H})$ are to be found. When Equation (5.27) is solvable, the algorithms proposed in Sections 4.3, 4.4 (or those in [49, 68]) can be employed to compute the solution. If $U_1,\ V_1$ cannot be found, D and $\hat{R}$ are not right zero Ω-coprime, and by Theorem 5.2, Equation (5.5) cannot be solved under the assumed conditions. If Equation (5.27) can be solved, in light of the proof of Theorem 5.2, $\hat{Q}$ and N must be left zero Ω-coprime. Then $\bar{U}_2,\ \bar{V}_2 \in \mathbf{M}(\mathbf{H})$ can be found for which the following left zero Ω-coprime equation holds.

$$\hat{Q}\bar{U}_2 + N\bar{V}_2 = I \tag{5.28}$$

Step 4. Calculate

$$\left.\begin{array}{l} U_2 = \bar{U}_2 - \hat{R}(V_1\bar{U}_2 - U_1\bar{V}_2) \\ V_2 = \bar{V}_2 + D(V_1\bar{U}_2 - U_1\bar{V}_2) \end{array}\right\} \tag{5.29}$$

such that the doubly zero Ω-coprime relation holds. i.e.

$$\begin{bmatrix} U_1 & V_1 \\ -N & \hat{Q} \end{bmatrix} \begin{bmatrix} D & -V_2 \\ \hat{R} & U_2 \end{bmatrix} = I \tag{5.30}$$

Then a solution to the skew Ω-prime equation (5.5) is given by

$$DU_1 + V_2 N = I.$$

□

It is easy to see that the skew Ω-prime equation (5.5) is equivalent to Equation (5.1) where $\Phi = \phi I$ and ϕ is a Ω-stable 2D polynomial. In addition, if we take $\Omega = \mathbf{C}^2$, then Equation (5.5) is equivalent to Equation (5.2). It is thus clear that these two problems can be solved by employing the procedure presented above.

The following theorem characterizes the general solutions to the skew Ω-prime equation (5.5) when D and N are square and nonsingular, and $|D|$ and $|N|$ are factor Ω-coprime.

Theorem 5.3 *If $\bar{U}$ and $\bar{V} \in \mathbf{M}(\mathbf{H})$ are solutions to Equation (5.5), and $|D|$ and $|N|$ are factor Ω-coprime, then*

$$\left.\begin{array}{l} U = \bar{U} + XN \\ V = \bar{V} - DX \end{array}\right\} \tag{5.31}$$

are the general solutions to Equation (5.5) for arbitrary $X \in \mathbf{M}(\mathbf{H})$.

Proof. First, it is trivial to verify that

$$\begin{aligned} DU + VN &= D(\bar{U} + XN) + (\bar{V} - DX)N \\ &= D\bar{U} + \bar{V}N + DXN - DXN \\ &= I \end{aligned}$$

Note that if U and $V \in \mathbf{M}(\mathbf{H})$ are also solution to Equation (5.5), the following relation is directly implied.

$$D(U - \bar{U}) = (\bar{V} - V)N \tag{5.32}$$

Since $|D|$ and $|N|$ are factor Ω-coprime, it follows that

$$D^{-1}(\bar{V} - V) = (U - \bar{U})N^{-1} = X \tag{5.33}$$

is in $\mathbf{M}(\mathbf{H})$, which implies Equation (5.31). The theorem has thus been established. □

5.4 Solution of General Bilateral 2D Polynomial Matrix Equation

Now, consider the bilateral 2D polynomial matrix equation (5.1) with a general Φ. For convenience, Equation (5.1) is rewritten here, i.e.,

$$DU + VN = \Phi \tag{5.34}$$

where D, N and $\Phi \in \mathbf{M}(\mathbf{R}[v,w])$ are given and U, $V \in \mathbf{M}(\mathbf{R}[v,w])$ are to be found. It might be noted that, by the approach of [108], the results presented in the previous sections can be employed to solve this equation. Noting that if D is square and nonsingular, the left MFD $D^{-1}\Phi$ can always be factored in the form

$$D^{-1}\Phi = \bar{\Phi}\bar{D}^{-1} \tag{5.35}$$

with $\bar{\Phi}$ and $\bar{D}$ right factor coprime, the following result can be established.

Theorem 5.4 *Consider the bilateral 2D polynomial matrix equation (5.34). If D is square and nonsingular and D and Φ are left zero coprime, then Equation (5.34) has a solution U, V if and only if $\bar{D}$ as given in Equation (5.35) and N are externally zero skew prime.*

Proof. The if part can be established as in [108].

For only if part, keeping in mind that D and Φ are left zero coprime from Equation (5.35) we have that $\det D = \det \bar{D}$, since zero coprimeness implies factor coprimeness. And in view of Theorem 2.4, $\bar{\Phi}$ and $\bar{D}$ must be right zero coprime as well. Moreover, when D and Φ are left zero coprime, one can find X, $Y \in \mathbf{M}(\mathbf{R}[v,w])$ such that $DX + \Phi Y = I$. It follows, in light of Equation (5.34), that

$$D(X + UY) + V(NY) = I \tag{5.36}$$

which directly implies that D and V are left zero coprime as well.

Then we have the representation

$$D^{-1}V = \tilde{V}\tilde{D}^{-1} \tag{5.37}$$

where $\tilde{V}$ and $\tilde{D}$ are right factor coprime. By Theorem 2.4 again, $\tilde{V}$ and $\tilde{D}$ are also right zero coprime. Then making use of Equations (5.35) and (5.37), we can rewrite Equation(5.34)

as

$$\tilde{V}\tilde{D}^{-1}N\bar{D} = \bar{\Phi} - U\bar{D} \tag{5.38}$$

According to Corollary 2.3, we conclude that

$$\tilde{D}^{-1}N\bar{D} = \bar{N} \tag{5.39}$$

is a 2D polynomial matrix. Then it follows that

$$U\bar{D} + \tilde{V}\bar{N} = \bar{\Phi} \tag{5.40}$$

Since $\bar{D}$ and $\bar{\Phi}$ are right zero coprime, there exist $\bar{X}$, $\bar{Y} \in \mathbf{M}(\mathbf{R}[v,w])$ such that $\bar{X}\bar{D} + \bar{Y}\bar{\Phi} = I$. In view of Equation (5.40), therefore, we obtain that

$$(\bar{X} + \bar{Y}U)\bar{D} + (\bar{Y}\tilde{V})\bar{N} = I \tag{5.41}$$

i.e., $\bar{D}$ and $\bar{N}$ are right zero coprime as well.

Since $\det \tilde{D} = \det \bar{D}$ from Equations (5.35) and (5.37), and by Corollary 2.2, we first have that $\tilde{D}$ and N in Equation (5.39) must be left factor coprime, and by Theorem 2.4 $\tilde{D}$ and N must also be left zero coprime.

Finally, after rewriting Equation (5.39) as

$$N\bar{D} = \tilde{D}\bar{N} \tag{5.42}$$

and using Theorem 5.1, it is concluded that $\bar{D}$ and N must be zero skew prime. The proof is thus completed. □

It might be noted from Theorem 5.4, that if only the sufficient condition for Equation (5.34) to be solvable is of interest, then the assumption that D and Φ are left zero coprime will not be necessary.

In addition, a necessary and sufficient condition for the general solution to Equation (5.34) can be given in the very same way as Theorem 5.3 under the assumptions that D and N are square and nonsingular and $|D|$ and $|N|$ are factor coprime.

5.5 Some More Properties on 2D Skew Ω-Prime Equation

We now derive a necessary and sufficient condition for the uniqueness of the skew complement pair [57] of a skew Ω-prime pair with respect to $\mathbf{H}$-unimodular equivalence.

Theorem 5.5 *Consider a pair, D and $N \in \mathbf{M}(\mathbf{H})$, of (skew Ω-prime) nonsingular matrices of equal order. If both $(\bar{D}, \bar{N})$ and $(\tilde{D}, \tilde{N})$ are the skew complement pairs of N and D , i.e., $ND = \bar{D}\bar{N} = \tilde{D}\tilde{N}$ with N and both $\bar{D}$ and $\tilde{D}$ left zero Ω-coprime, and D and both $\bar{N}$ and $\tilde{N}$ right zero Ω-coprime, or equivalently,*

$$\begin{bmatrix} D & Y \\ -\bar{N} & U \end{bmatrix} \begin{bmatrix} X & -V \\ N & \bar{D} \end{bmatrix} = \begin{bmatrix} D & \tilde{Y} \\ -\tilde{N} & \tilde{U} \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{V} \\ N & \tilde{D} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{5.43}$$

with X, Y, U, V, and $\tilde{X}, \tilde{Y}, \tilde{U}, \tilde{V} \in \mathbf{M}(\mathbf{H})$, then $\tilde{N} = W\bar{N}, \tilde{D}W = \bar{D}$ for some $\mathbf{H}$-unimodular matrix W if and only if $\tilde{X} = X + TN,\ \tilde{Y} = Y - DT$ for some $T \in \mathbf{M}(\mathbf{H})$.

Proof: The sufficiency can be proved in a similar way as in [108], hence is omitted. We now show the necessity. Suppose $\tilde{N} = W\bar{N}, \tilde{D}W = \bar{D}$ holds for some $\mathbf{H}$-unimodular matrix W. Then in light of Equation (5.43), it follows that

$$\begin{bmatrix} X & -V \\ N & \bar{D} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} D & Y \\ -\bar{N} & U \end{bmatrix}$$

$$= \begin{bmatrix} X & -VW^{-1} \\ N & \tilde{D} \end{bmatrix} \begin{bmatrix} D & Y \\ -\tilde{N} & WU \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.44}$$

Then, Equations (5.43) and (5.44) imply that

$$\begin{bmatrix} \tilde{X} & -\tilde{V} \\ N & \tilde{D} \end{bmatrix} \begin{bmatrix} D & Y \\ -\tilde{N} & WU \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \tag{5.45}$$

with $T = \tilde{X}Y - \tilde{V}WU \in \mathbf{M}(\mathbf{H})$. Postmultiplying Equation (5.45) by

$$\begin{bmatrix} X & -VW^{-1} \\ N & \tilde{D} \end{bmatrix} = \begin{bmatrix} D & Y \\ -\tilde{N} & WU \end{bmatrix}^{-1} \tag{5.46}$$

yields that $\tilde{X} = X + TN$. Then premultiplying Equation (5.45) by

$$\begin{bmatrix} D & \tilde{Y} \\ -\tilde{N} & \tilde{U} \end{bmatrix} = \begin{bmatrix} \tilde{X} & -\tilde{V} \\ N & \tilde{D} \end{bmatrix}^{-1} \tag{5.47}$$

we obtain $\tilde{Y} = Y - DT$. □

Corollary 5.1 *Let D, N be nonsingular matrices in $\mathbf{M}(\mathbf{H})$, with $\det D$ and $\det N$ being factor Ω-coprime. If both $(\bar{D}, \bar{N})$ and $(\tilde{D}, \tilde{N})$ are the skew complement pairs of N and D, then $\tilde{N} = W\bar{N}$, $\tilde{D}W = \bar{D}$ for some $\mathbf{H}$-unimodular matrix W.*

Proof: In light of Theorem 4.7, Equation (5.43) equivalently holds under the assumed conditions, which directly implies $DX + YN = D\tilde{X} + \tilde{Y}N = I$ with $\tilde{X} = X + TN$ and $\tilde{Y} = Y - DT$, by Theorem 5.3, for some $T \in \mathbf{M}(\mathbf{H})$. Then the corollary directly follows from Theorem 5.5. □

Next we consider whether some new solvability condition of Equation (5.5) can be established by utilizing only the matrices D and N. This is motivated by a conjecture given by Emre [33] on the "fixed poles" for the skew prime equation over $\mathbf{R}(z)$. If the conjecture is true and further it also holds for the 2D case, it might be possible to test the solvability of Equation (5.5) by checking whether these "fixed poles" lie in Ω. The problem considered in [33] can be restated in a simplified fashion as follows. Consider, without loss of generality, the skew prime equation

$$D(z)X(z) + Y(z)N(z) = I \tag{5.48}$$

where D, $N \in \mathbf{M}(\mathbf{R}[z])$ with D nonsingular, and $X, Y \in \mathbf{M}(\mathbf{R}(z))$. Equation (5.48) is equivalent to

$$D(z)\bar{X}(z) + \bar{Y}(z)N(z) = t(z)I \tag{5.49}$$

where $\bar{X}, \bar{Y} \in \mathbf{M}(\mathbf{R}[z])$ and $t(z) \in \mathbf{R}[z]$. Suppose in Equation (5.49) N has linearly independent columns over $\mathbf{R}[z]$. Let $\gamma(z)$ be the product of all the invariant factors of N, and $\lambda(z) = \det D(z)$. Let $\lambda_1(z)$ be the greatest common factor of $\gamma(z)$ and $\lambda(z)$. Emre [33] then gave the conjecture that the "fixed poles" of Equation (5.48) are the roots of $\lambda_1(z)$. We show here that this is not true by a simple counterexample. Assume in Equation (5.49),

$$D = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, \qquad N = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.50}$$

Since $\det D = \det N = z$, $\lambda_1(z) = z$. However, it is trivial to verify that

$$\bar{X} = \begin{bmatrix} 1-z & z \\ z^2 & 1 \end{bmatrix}, \qquad \bar{Y} = \begin{bmatrix} 1 & -z \\ -z^2 & 1-z \end{bmatrix} \tag{5.51}$$

satisfy Equation (5.49) with $t(z) = 1$. So in fact it is not necessary for λ_1 to be the factor of $t(z)$, i.e., the roots of $\lambda_1(z)$ are not the "fixed poles" of Equation (5.48). As a matter of fact, the counterexample we have given in Section 5.1 shows the same consequence for 2D skew Ω-prime equation. Therefore, we finally arrived at a negative conclusion for the above attempt.

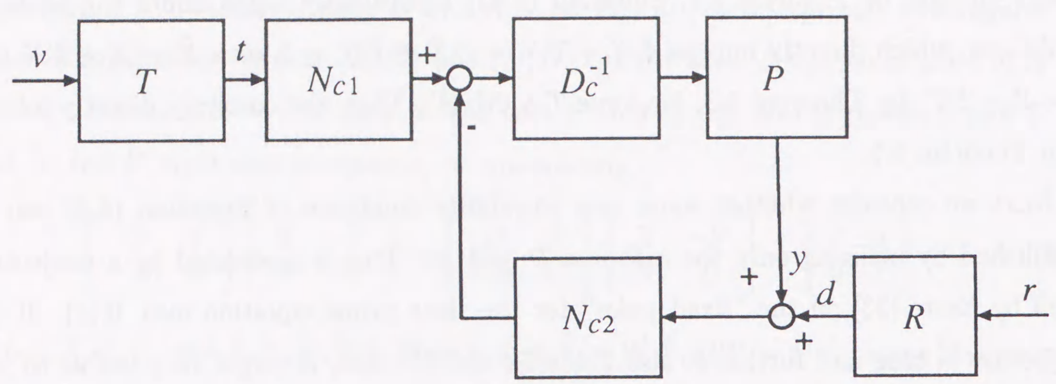


Figure 5.1: Configuration for 2D Tracking and Disturbance Rejection

5.6 Asymptotic and Deadbeat Tracking and Disturbance Rejecting Problems of 2D Systems

Consider the configuration shown in Figure 5.1, where P is a given plant, and T and R are specified reference signal generators. The asymptotic tracking problem and the disturbance rejection problem of the 2D system of Figure 5.1 can be formulated as to find a stabilizing compensator C that achieves correspondingly one of the following additional features:

(i) Asymptotic Tracking: $(I-H_{yt})T \in \mathbf{M}(\mathbf{H})$, i.e., the transfer matrix from v to $e=t-y$ belongs to $\mathbf{M}(\mathbf{H})$;

(ii) Asymptotic Disturbance Rejection: $H_{yd}R \in \mathbf{M}(\mathbf{H})$, i.e., the transfer matrix from r to y belongs to $\mathbf{M}(\mathbf{H})$.

Before proceeding, the following result can be easily given in a similar way to the 1D case [103].

Lemma 5.1 *Suppose $E,\ K,\ F \in \mathbf{M}(\mathbf{H})$, $L = EK^{-1}F$, and $K,\ F$ left zero Ω-coprime. Then $L \in \mathbf{M}(\mathbf{H})$ if and only if $EK^{-1} \in \mathbf{M}(\mathbf{H})$.*

Proof. The sufficiency is obvious. To show the necessity, choose $X,\ Y \in \mathbf{M}(\mathbf{H})$ such that $KX+FY=I$, then we have

$$K^{-1} = X + K^{-1}FY \tag{5.52}$$

and

$$EK^{-1} = EX + EK^{-1}FY = EX + LY. \tag{5.53}$$

Therefore, if $L \in \mathbf{M}(\mathbf{H})$ then $EK^{-1} \in \mathbf{M}(\mathbf{H})$. □

Now, the necessary and sufficient conditions and solutions to the two problems will be given respectively as follows.

Solution of (i): According to the result of Theorem 4.9, we have that

$$H_{yt} = N_pQ \tag{5.54}$$

Suppose that the left zero Ω-coprime MFD of T exists, say $T = D_t^{-1}N_t$. Then the tracking problem has solution if and only if

$$(I-N_pQ)D_t^{-1}N_t \in \mathbf{M}(\mathbf{H}) \tag{5.55}$$

By Lemma 5.1, this is true if and only if

$$(I-N_pQ)D_t^{-1} = W \tag{5.56}$$

for some $W \in \mathbf{M}(\mathbf{H})$.

So we obtain a skew Ω-coprime equation as

$$N_pQ + WD_t = I \tag{5.57}$$

Theorem 5.6 *Suppose that plant P is given as in Theorem 4.9, i.e., $P \in \mathbf{M}(\mathbf{G})$ and $N_pD_p^{-1}$, $\tilde{D}_p^{-1}\tilde{N}_p$ are right and left zero Ω-coprime MFD of P. Further, suppose that $D_t^{-1}N_t$ is the left zero Ω-coprime MFD of the given reference signal generator $T \in \mathbf{M}(\mathbf{G})$. Then the asymptotic tracking problem of 2D system in Figure 5.1 has solution if and only if N_p and D_t are skew Ω-prime.*

The proof is obvious from the above discussions. To solve the equation (5.57), the results presented in the previous sections can be applied.

Solution to (ii): Suppose that the left zero Ω-coprime MFD of R exists, say $R = D_r^{-1}N_r$, and from Theorem 4.9 we have

$$H_{yd} = -N_p(V - S\tilde{D}_p) \tag{5.58}$$

then we get

$$H_{yd}R = -N_p(V - S\tilde{D}_p)D_r^{-1}N_r \in \mathbf{M}(\mathbf{H}) \tag{5.59}$$

By Lemma 5.1, this is true if and only if

$$-N_p(V - S\tilde{D}_p)D_r^{-1} = Z \in \mathbf{M}(\mathbf{H}) \tag{5.60}$$

namely,

$$-N_pS\tilde{D}_p + ZD_r = N_pV \tag{5.61}$$

Therefore it is clear that the problem is solvable if and only if there exist S and $Z \in \mathbf{M}(\mathbf{H})$ such that Equation (5.61) holds.

Theorem 5.7 *Equation (5.61) has solution if and only if $\tilde{D}_p$ and D_r are right zero Ω-coprime.*

Proof: Sufficiency. If $\tilde{D}_p$ and D_r are right zero Ω-coprime then there exist X and $Y \in \mathbf{M}(\mathbf{H})$ such that

$$X\tilde{D}_p + YD_r = I \tag{5.62}$$

Then

$$\begin{aligned} N_pV &= N_pV(X\tilde{D}_p + YD_r) \\ &= N_p(VX)\tilde{D}_p + (N_pVY)D_r \\ &= -N_pS_0\tilde{D}_p + Z_0D_r \end{aligned} \tag{5.63}$$

where $S_0 = -VX$, $Z_0 = N_pVY$ is a special solution to Equation (5.61).

Moreover, we can find

$$D_r\tilde{D}_p^{-1} = F_1^{-1}F_2 \tag{5.64}$$

$$F_1D_r = F_2\tilde{D}_p \tag{5.65}$$

Then

$$-N_p(WF_2)\tilde{D}_p + (N_pWF_1)D_r = 0 \tag{5.66}$$

for arbitrary $W \in \mathbf{M}(\mathbf{H})$.

So the general solution can be given as

$$S = S_0 + WF_2 \tag{5.67}$$

$$Z = Z_0 + N_pWF_1 \tag{5.68}$$

Necessity. Noting that, as shown in Theorem 4.10, U, V satisfy the right zero Ω-coprime equation $UD_p + VN_p = I$, we can find U_l, $V_l \in \mathbf{M}(\mathbf{H})$, in view of Theorem 4.7, such that the doubly Ω-coprime relation holds, i.e.,

$$\begin{bmatrix} U & V \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix}\begin{bmatrix} D_p & -V_l \\ N_p & U_l \end{bmatrix} = I. \tag{5.69}$$

Then it follows that

$$\begin{bmatrix} D_p & -V_l \\ N_p & U_l \end{bmatrix}\begin{bmatrix} U & V \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} = I. \tag{5.70}$$

Therefore,

$$N_pV + U_l\tilde{D}_p = I. \tag{5.71}$$

Then from Equation (5.61),

$$-N_pS\tilde{D}_p + ZD_r = I - U_l\tilde{D}_p, \tag{5.72}$$

$$(U_l - N_pS)\tilde{D}_p + (Z)D_r = I. \tag{5.73}$$

Therefore, $\tilde{D}_p$ and D_r are right Ω-coprime. □

On the other hand, the deadbeat tracking and deadbeat disturbance rejection problems for the system of Figure 5.1 can be formulated as:

(i) Deadbeat tracking: $(I - H_{yt})T \in \mathbf{M}(\mathbf{R}[z,w])$;

(ii) Deadbeat disturbance rejection: $H_{yd}R \in \mathbf{M}(\mathbf{R}[z,w])$.

To solve these problems, we can just substitute the condition $\Omega = \mathbf{C}^2$ into the previous results for asymptotic problems. For this case, zero Ω-coprimeness restricts to zero coprimeness, and all the other arguments apply directly. Hence, the details are omitted here.

5.7 Example

Now we use the above procedure to show that N and D given below are skew Ω-prime. Here Ω is taken as closed unit bidisc, i.e., $\Omega = \bar{\mathbb{U}}^2$. Let

$$D = \begin{bmatrix} 2(2v-3)(1.6w+2v-3) & 0 \\ 0 & 2(2v-1) \end{bmatrix}$$

$$N = \begin{bmatrix} 4(2v-1) & (2v-3)^3 \\ 3.2w(1.6w-2)+(2v-1) & 0.8w\{(1.6w-2)(4v2-16v+19)+8\} \end{bmatrix}$$

If $\Delta = \det D$, then Equation (5.17) is satisfied with

$$D^{+} = \begin{bmatrix} 2(2v-1) & 0 \\ 0 & 2(2v-3)(1.6w+2v-3) \end{bmatrix}$$

$$Q = \begin{bmatrix} 2(1.6w+2v-3) & 0 \\ -1.6w(1.6w-2)(2v-3) & 2(2v-3)(2v-1) \end{bmatrix}$$

$$R = \begin{bmatrix} 4(1.6w+2v-3) & (2v-3)^2 \\ (2v-1)(2v-3) & 6.4w \end{bmatrix}$$

It is easy to verify that

$$\bar{Q} = \Delta Q^{-1} = \begin{bmatrix} 2(2v-1)(2v-3) & 0 \\ 1.6w(1.6w-2)(2v-3) & 2(1.6w+2v-3) \end{bmatrix}$$

Since $\det \bar{Q} = \Delta$, we have that $\hat{G} = I$ and $\hat{Q} = \bar{Q}$, $\bar{R} = R$.

According to [68], it can be shown that D and $\hat{R}$ are right zero Ω-coprime, i.e., there exist U_1, $V_1 \in \mathbf{M}(\mathbf{H})$ as

$$U_1 = \begin{bmatrix} \dfrac{64w^2+40w+25v+25}{30(2v-3)(4w-5)} & \dfrac{w(32wv^2-128wv+152w+20v^2-80v+135)}{15(2v-3)(4w-5)} \\ \dfrac{v}{2} & \dfrac{4v^3-16v^2+19v-4}{8} \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -\dfrac{2w(8w+5)}{15(4w-5)} & -\dfrac{4w+5v}{3(2v-3)(4w-5)} \\ -\dfrac{v(2v-3)}{4} & 0 \end{bmatrix}$$

such that Equation (5.27) holds.

Then by Theorem 5.2 N and $\hat{Q}$ must be left zero Ω-coprime. Using the method of

Section 4.3 or [68], we can find U_2, $V_2 \in \mathbf{M}(\mathbf{H})$ in Equation (5.28) as

$$\bar{U}_2 = \begin{bmatrix} -\dfrac{4vw-26w-10v^2+25v-10}{4(2v-3)(4w-5)} & \dfrac{2(4w+5v)}{3(2v-3)(4w-5)} \\ -\dfrac{w(8w-10v-15)}{5(2v-3)(4w-5)} & \dfrac{5(v+1)}{6(4w-5)} \end{bmatrix}$$

$$\bar{V}_2 = \begin{bmatrix} -\dfrac{2w}{4w-5} & -\dfrac{4w+5v}{3(4w-5)} \\ \dfrac{2w-5v}{2(2v-3)(4w-5)} & 0 \end{bmatrix}$$

Then the doubly zero Ω-coprime relation in Equation (5.30) can be obtained with

$$V_2 = \bar{V}_2 + D(V_1\bar{U}_2 - U_1\bar{V}_2) = \begin{bmatrix} -\dfrac{2w}{4w-5} & \dfrac{4w+5v}{3(4w-5)} \\ -\dfrac{v}{4} & 0 \end{bmatrix}$$

$$U_2 = \bar{U}_2 - \hat{R}(V_1\bar{U}_2 - U_1\bar{V}_2) = \begin{bmatrix} u_2^{(1,1)}(v,w) & \dfrac{2(4w+5v)}{3(2v-3)(4w-5)} \\ \dfrac{w(8w-5)}{5(4w-5)} & \dfrac{5(v+1)}{6(4w-5)} \end{bmatrix}$$

where

$$u_2^{(1,1)}(v,w) = \frac{16v^3w-64v^2w+76vw+16w-20v^3+80v^2-95v+20}{8(2v-3)(4w-5)}$$

It is clearly implied that Equation (5.5) possess solution U_1 and $V_2 \in \mathbf{M}(\mathbf{H})$. This shows that the given D and N are skew Ω-prime.

5.8 Summary

For the motivation of deadbeat and asymptotic servo control questions of 2D systems, two special situations of the bilateral 2D polynomial matrix equation $DU + VN = \Phi$ when $\Phi = I$ and $\Phi = \phi I$ with ϕ a Ω-stable 2D polynomial have been first considered. Based on the generalization of the concepts of factor coprimeness, zero coprimeness and zero skew primeness in the 2D polynomial ring to the ring of causal Ω-stable 2D rational functions as, respectively, factor Ω-coprimeness, zero Ω-coprimeness and (externally zero) skew Ω-primness, a unified constructive solution procedure to the two problems mentioned above has been given. The presented procedure applies under a less restrictive condition than the known one [94] when the equation is considered in the ring of 2D polynomials. The general solutions to the problems have been investigated as well. Although it has not been stated in the previous sections, it might be remarked here that minimum degree solutions can be constructed from the general one by employing the method of [94] since the condition that D is monic (in e.g. w) can always be accomplished when D is square and invertible.

Based on the results above, the applicable solvability conditions to the bilateral equation (5.1) with Φ being a general 2D polynomial matrix have also been obtained.

Some results on the uniqueness of the skew complement pair of a pair of skew Ω-prime matrices with respect to $\mathbf{H}$-unimodular equivalence are derived. The result of Corollary 5.1 may be viewed as an extension of the result of [108]. It has been shown in [57] that the condition of [108], i.e., the coprimeness of $\det D$ and $\det N$, is also a necessary one for the uniqueness of the 1D skew complement pair. However, the necessity of Corollary 5.1 remains to be proved.

Moreover, a counterexample has been given for Emre's conjecture on the "fixed poles" of the skew prime equation over $\mathbf{R}(z)$.

By making direct use of the obtained results for skew Ω-prime equation, the asymptotic and deadbeat output regulation and tracking questions of 2D systems [52, 90, 112] have been solved in a unified way.

Finally, the proposed procedure has been illustrated by an example.

Chapter 6

Practical-Stabilization of nD Systems by MFD Approach

6.1 Introduction

In many practical situations of nD signal processing, such as seismic and image processing, the independent variables $i_1, \ldots, i_n$ of an nD signal $x(i_1, \ldots, i_n)$ are usually bounded spatial variables, except that perhaps one is the unbounded temporal variable. Taking this feature into account, Agathoklis and Bruton [1] developed the concept of *practical-BIBO stability* for nD discrete systems, and showed that the conventional-BIBO stability conditions are too restrictive for many applications.

For designing practical-BIBO stable nD digital filters, some results have been documented in the literature (see, e.g., [80]). However, for feedback practical-stabilization of nD systems, a general effective method is not yet available. Since practical-BIBO stability conditions are much weaker than conventional ones, there exist systems that are practical-BIBO stable but not conventional-BIBO stable [1]. This fact means that, to design such practical-BIBO stable feedback systems, the existing methods developed under the conventional-BIBO stability concept cannot be applied. This is because that the current design methods of nD ($n = 2$) system are mainly based on the idea of making the closed-loop systems (denominator) separable [54] by using local state or output feedback, and obviously the separable structure precludes the practical-BIBO stable situation mentioned above.

The objective of this chapter and the next chapter is to consider the feedback stabilization and some other fundamental control problems of nD systems in the practical sense of

[1]. Namely, the input and output signals are assumed to be unbounded in, at most, one dimension. From now on, an MIMO nD feedback system is said to be practically-stable if every entry of its closed-loop transfer matrix corresponds to a SISO practically-BIBO stable system. By practical-stabilization, then, we mean that to make a given system practically stable by feedback scheme. In particular, this chapter deals with, by using algebraic approach, the problem of feedback practical-stabilization of nD discrete systems. In Section 6.2, we briefly review the concept of practical-BIBO stability for nD discrete systems and the necessary and sufficient conditions shown by Agathoklis and Bruton [1]. It will be seen that the practical-BIBO stability of an nD system is in fact equivalent to the stabilities of n 1D systems, so it is much weaker than the conventional-BIBO stability. In Section 6.3, a constructive algorithm is presented for solving Bezout equation over the ring of practically-stable rational functions. Section 6.4 is devoted to the problems of feedback practical-stabilization of nD systems and parametrization of all nD practically-stabilizing compensators, and Section 6.5 gives some illustrative examples.

6.2 Practical-BIBO Stability for nD Discrete Systems

Consider the class of (SISO) causal LSI nD discrete systems for which the input $u(i_1, \ldots, i_n)$ and the output $y(i_1, \ldots, i_n)$ are related by the nD convolution sum:

$$y(i_1, \ldots, i_n) = \sum_{k_1=0}^{i_1} \cdots \sum_{k_n=0}^{i_n} h(i_1 - k_1, \ldots, i_n - k_n) u(k_1, \ldots, k_n) \qquad (6.1)$$

where $h(i_1, \ldots, i_n)$ is the impulse response. By using nD z-transform we can obtain the transfer function of nD system (6.1).

For an nD system, the conventional-BIBO stability is defined by Definition 3.2. In view of the convenience of comparison it with practical-BIBO stability, we rewrite the definition here.

Definition 6.1 [50, 51]

An nD system is BIBO stable if and only if, for all input signals $u(i_1, \ldots, i_n)$ such that

$$|u(i_1, \ldots, i_n)| \le M < \infty \quad \forall (i_1, \ldots, i_n) \in Z_+^n \qquad (6.2)$$

where M is a finite real number, there exists a finite real number L such that, for the output $y(i_1, \ldots, i_n)$ of the system,

$$|y(i_1, \ldots, i_n)| \le L < \infty \qquad (6.3)$$

holds.

On the other hand, the practical-BIBO stability introduced by Agathoklis and Bruton [1] is defined by:

Definition 6.2 [1]

An nD system is practical-BIBO stable if and only if, for all input signals $u(i_1, \ldots, i_n)$ such that

$$|u(i_1, \ldots, i_n)| \le M < \infty \quad \forall (i_1, \ldots, i_n) \in Z_+^{-n} \qquad (6.4)$$

where M is a finite real number, there exists a finite real number L such that, for the output of the system $y(i_1, \ldots, i_n)$, the relation

$$|y(i_1, \ldots, i_n)| \le L < \infty \qquad (6.5)$$

is satisfied.

The difference between these two definitions is that, for the case of practical-BIBO stability, the behaviour of the system at the points where more than one of the indeterminates take infinite value are not considered (see [1]). Since, from a practical point of view, we are not interested in input signals that are spatially unbounded in more than one dimension, the behaviour of a system at these points is of no practical interest.

As stated in Section 3.5, a well-known necessary and sufficient condition for the conventional-BIBO stability is that the impulse response of the system satisfies the following relation [51]:

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} |h(i_1, i_2, \ldots, i_n)| < \infty. \qquad (6.6)$$

In contrast, Agathoklis and Bruton [1] have shown the following theorem which reveals the relationship between the practical-BIBO stability and the impulse response of an nD system.

Theorem 6.1 [1]

An nD discrete system is practically-BIBO stable if and only if the n inequalities are satisfied:

$$\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_k=0}^{N_k=\infty}\cdots\sum_{i_n=0}^{N_n}|h(i_1,i_2,\ldots,i_k,\ldots,i_n)|<\infty,\quad k=1,2,\ldots,n \tag{6.7}$$

where $N_1,\ N_2,\ldots,N_{k-1},\ N_{k+1},\ldots,N_n$ *are any finite integers.*

The following theorem relates the practical-BIBO stability to the singularities of the transfer function of an nD system:

$$h(z_1,\ldots,z_n)=\frac{n(z_1,\ldots,z_n)}{d(z_1,\ldots,z_n)} \tag{6.8}$$

Theorem 6.2 [1]

nD system (6.8) is practically-BIBO stable if and only if

$$d(0,\ldots,z_k,\ldots,0)\neq 0$$

$$\forall z_k\in\bar{U}=\{z\in\mathbf{C}|\ |z|\le 1\},\ \ k=1,2,\ldots,n \tag{6.9}$$

As shown in Section 3.5, for an nD system given by transfer function (6.8), if it has no nonessential singularity of the second kind [20] on T^2 when $n=2$ [46], or on $\bar{\mathrm{U}}^n-\mathrm{U}^n$ when $n>2$ [98], the system is BIBO stable in the conventional sense if and only if

$$d(z_1,\ldots,z_n)\neq 0\quad \forall(z_1,\ldots,z_n)\in\bar{\mathrm{U}}^n. \tag{6.10}$$

It is then obvious that the condition (6.9) for practical-BIBO stability is in fact equivalent to the stabilities of n 1D systems, and this is much weaker than the condition (6.10) for conventional-BIBO stability.

6.3 Solution for Bezout Equation over the Ring of Practically-Stable Rational Functions $\tilde{\mathbf{H}}$

Throughout the remainder of the thesis, we call an nD rational function practically-stable if its denominator satisfies the condition (6.9). Let $\mathbf{G}$ be the ring of nD causal rational function, $\tilde{\mathbf{H}}$ be the ring of nD practically-stable rational function, i.e.,

$$\mathbf{G}=\{n/d\mid n,d\in\mathbf{R}[z_1,\ldots,z_n],\ d(0,\ldots,0)\neq 0\},$$

$$\tilde{\mathbf{H}}=\{n/d\in\mathbf{G}\mid d(0,\ldots,z_k,\ldots,0)\neq 0\ \forall z_k\in\bar{\mathrm{U}},\ k=1,2,\ldots,n\},$$

and let

$$\tilde{\mathbf{I}}=\{h\in\tilde{\mathbf{H}}\mid h^{-1}\in\mathbf{G}\},$$

$$\tilde{\mathbf{J}}=\{h\in\tilde{\mathbf{H}}\mid h^{-1}\in\tilde{\mathbf{H}}\}.$$

Then an element of $\mathbf{M}(\tilde{\mathbf{H}})$ is said to be $\mathbf{G}$-unimodular (respectively $\tilde{\mathbf{H}}$-unimodular) if and only if it is square and its determinant belongs to $\tilde{\mathbf{I}}$ ($\tilde{\mathbf{J}}$). If $P\in\mathbf{M}(\mathbf{G})$ can be written as $P=N_pD_p^{-1}$, where $D_p,\ N_p\in\mathbf{M}(\tilde{\mathbf{H}})$ and D_p is $\mathbf{G}$-unimodular, we refer to such $N_pD_p^{-1}$ as a right MFD of P (on $\{\mathbf{G},\tilde{\mathbf{H}},\tilde{\mathbf{I}},\tilde{\mathbf{J}}\}$).

Definition 6.3 *With respect to right MFD* $N_pD_p^{-1}$, *if there exist* $U,V\in\mathbf{M}(\tilde{\mathbf{H}})$ *such that the Bezout equation*

$$UD_p+VN_p=I \tag{6.11}$$

holds, then and only then, we say that N_p *and* D_p *are right coprime on* $\tilde{\mathbf{H}}$ *and that* $N_pD_p^{-1}$ *is a right coprime MFD on* $\tilde{\mathbf{H}}$.

The dual definitions on left are given analogously. It is easy to see that for any $P\in\mathbf{M}(\mathbf{G})$, we can always find $N_p,\ D_p\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])\subset\mathbf{M}(\tilde{\mathbf{H}})$ such that $P=N_pD_p^{-1}$, but N_p and D_p are not in general right coprime on $\tilde{\mathbf{H}}$. As a main result of this section, the following theorem shows a necessary and sufficient condition for the existence of right coprime MFD of P on $\tilde{\mathbf{H}}$. Suppose, without loss of generality, $N_p,\ D_p\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$. Let $\mathcal{I}_k$ denote the ideal generated by the all maximal minors of the matrix

$$\begin{bmatrix} D_p(0,\ldots,z_k,\ldots,0)\\ N_p(0,\ldots,z_k,\ldots,0)\end{bmatrix} \tag{6.12}$$

and $\mathcal{V}(\mathcal{I}_k)$ be the algebraic variety of $\mathcal{I}_k$, where $k=1,2,\ldots,n$.

Theorem 6.3 *For* $N_pD_p^{-1}$ *where* $N_p,\ D_p\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$, D_p *and* N_p *is right coprime on* $\tilde{\mathbf{H}}$ *if and only if*

$$\mathcal{V}(\mathcal{I}_k)\cap\bar{U}=\emptyset,\qquad k=1,2,\ldots,n \tag{6.13}$$

Proof. The solvability of (6.11) is obviously equivalent to

$$XD_p+YN_p=\Phi \tag{6.14}$$

where $X,\ Y,\ \Phi\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$ and $\det\Phi\in\tilde{\mathbf{J}}$.

Then the necessity can be easily shown by invoking Cauchy-Binet Theorem. Namely, if $z_k^0 \in \mathcal{V}(\mathcal{I}_k)$, z_k^0 is also a zero of

$$\det \Phi(0,\ldots,z_k,\ldots,0) =$$

$$\det\left\{[X(0,\ldots,z_k,\ldots,0)\; Y(0,\ldots,z_k,\ldots,0)]\begin{bmatrix} D_p(0,\ldots,z_k,\ldots,0) \\ N_p(0,\ldots,z_k,\ldots,0)\end{bmatrix}\right\} \quad (6.15)$$

Then it is trivial to see that $\det\Phi \notin \tilde{\mathbf{J}}$, i.e., Equation (6.14) can never be satisfied, if $z_k^0 \in \bar{U}$.

So we have only to show the sufficiency. If the condition (6.13) is satisfied, the following 1D polynomial matrix equations can be solved by known 1D methods [62].

$$\tilde{X}_k(z_k)D_p(0,\ldots,z_k,\ldots,0) + \tilde{Y}_k(z_k)N_p(0,\ldots,z_k,\ldots,0) = \tilde{\Phi}_k(z_k) \quad (6.16)$$

where $\tilde{X}_k(z_k)$, $\tilde{Y}_k(z_k) \in \mathbf{M}(\mathbf{R}[z_k])$, and

$$\det\tilde{\Phi}_k(z_k) \neq 0 \quad \forall z_k \in \bar{\mathrm{U}}, \quad k = 1,2,\ldots,n \quad (6.17)$$

The general solution for Equation (6.16) is given in the form

$$\begin{cases} \bar{X}_k(z_k) = \tilde{X}_k(z_k) + R_k(z_k)\tilde{N}_k(z_k) \\ \bar{Y}_k(z_k) = \tilde{Y}_k(z_k) - R_k(z_k)\tilde{D}_k(z_k)\end{cases} \quad (6.18)$$

where $R_k(z_k) \in \mathbf{M}(\mathbf{R}[z_k])$ is arbitrary, $\tilde{D}_k(z_k)$, $\tilde{N}_k(z_k)$ satisfy the following relation and are left coprime on $\mathbf{R}[z_k]$.

$$\tilde{D}_k^{-1}(z_k)\tilde{N}_k(z_k) = N_p(0,\ldots,z_k,\ldots,0)D_p^{-1}(0,\ldots,z_k,\ldots,0) \quad (6.19)$$

Suppose $E_k(z_k)$ is the right greatest common factor of $D_p(0,\ldots,z_k,\ldots,0)$ and $N_p(0,\ldots,z_k,\ldots,0)$, then it follows that

$$\det D_p(0,\ldots,z_k,\ldots,0) = \det\tilde{D}_k(z_k)\det E_k(z_k). \quad (6.20)$$

Since $\det D_p \in \tilde{\mathbf{I}}$, i.e., $\det D_p(0,\ldots,0) \neq 0$, we see that

$$\det\tilde{D}_k(0) \neq 0. \quad (6.21)$$

So we can set:

$$R_k(z_k) = \tilde{Y}_k(0)\tilde{D}_k^{-1}(0) \triangleq R_k \in \mathbf{M}(\mathbf{R}) \quad (6.22)$$

Substituting R_k into (6.18) and using (6.16), (6.17) we see that

$$\det\bar{X}_k(0) = \det\{\tilde{\Phi}_k(0)D_p^{-1}(0,\ldots,0)\} \neq 0 \quad (6.23)$$

$$\bar{Y}_k(0) = 0 \quad (6.24)$$

On the other hand, we can write $\bar{X}_k(z_k)$ as the sum

$$\bar{X}_k(z_k) \triangleq \hat{X}_k(z_k) + \bar{X}_k(0) \quad (6.25)$$

where $\bar{X}_k(0)$ corresponds to the constant terms of $\bar{X}(z_k)$, and $\hat{X}_k(z_k)$ denotes all the other terms which involve the variable z_k. Obviously,

$$\hat{X}_k(0) = 0 \quad (6.26)$$

According to Equation (6.23), $\bar{X}_k^{-1}(0)$ exists. Substituting $\bar{X}_k(z_k)$, $\bar{Y}_k(z_k)$ into (6.16) and then premultiplying it by $\bar{X}_k^{-1}(0)$ yield the result

$$X_k(z_k)D_p(0,\ldots,z_k,\ldots,0) + Y_k(z_k)N_p(0,\ldots,z_k,\ldots,0) = \Phi_k(z_k) \quad (6.27)$$

where

$$X_k(z_k) = \bar{X}_k^{-1}(0)\bar{X}_k(z_k) = \bar{X}_k^{-1}(0)\hat{X}_k(z_k) + I$$

$$\triangleq X_k'(z_k) + I \quad (6.28)$$

$$Y_k(z_k) = \bar{X}_k^{-1}(0)\bar{Y}_k(z_k) \quad (6.29)$$

$$\Phi_k(z_k) = \bar{X}_k^{-1}(0)\tilde{\Phi}_k(z_k) \quad (6.30)$$

and

$$X_k'(0) = 0, \quad (6.31)$$

$$Y_k(0) = 0, \quad (6.32)$$

$$\det\Phi_k(z_k) \neq 0 \;\forall z_k \in \bar{\mathrm{U}}. \quad (6.33)$$

By the above results, then, the solution to (6.14) can be constructed as follows.

$$X(z_1,\ldots,z_n)=\sum_{k=1}^{n}X_k'(z_k)+I \tag{6.34}$$

$$Y(z_1,\ldots,z_n)=\sum_{k=1}^{n}Y_k(z_k) \tag{6.35}$$

$\Phi(z_1,\ldots,z_n)$ is obtained by the computation

$$\begin{aligned}\Phi(z_1,\ldots,z_n) &= \{\sum_{k=1}^{n}X_k'(z_k)+I\ \}\ D_p(z_1,\ldots,z_n)+\\ &\quad +\{\sum_{k=1}^{n}Y_k(z_k)\ \}\ N_p(z_1,\ldots,z_n)\end{aligned} \tag{6.36}$$

In view of Equations (6.27)$\sim$(6.35), it follows the relation

$$\begin{aligned}&\Phi(0,\ldots,z_k,\ldots,0)=\\ &= \{\sum_{\substack{j=1\\ j\neq k}}^{n}X_j'(0)+X_k'(z_k)+I\ \}\ D_p(0,\ldots,z_k,\ldots,0)\\ &\quad +\{\sum_{\substack{j=1\\ j\neq k}}^{n}Y_j(0)+Y_k(z_k)\ \}\ N_p(0,\ldots,z_k,\ldots,0)\\ &= X_k(z_k)D_p(0,\ldots,z_k,\ldots,0)+\\ &\quad +Y_k(z_k)N_p(0,\ldots,z_k,\ldots,0)\\ &= \Phi_k(z_k)\end{aligned} \tag{6.37}$$

which shows that

$$\det\Phi(0,\ldots,z_k,\ldots,0)=\det\Phi_k(z_k)\neq 0$$

$$\forall z_k\in\bar{\mathrm{U}},\ \ k=1,\ldots,n \tag{6.38}$$

i.e., $\det\Phi\in\tilde{\mathrm{J}}$. Therefore, the solution to (6.11) directly reads as

$$\begin{cases}U=\Phi^{-1}X\\ V=\Phi^{-1}Y\end{cases} \tag{6.39}$$

□

Using the right and left coprime MFD's on $\tilde{\mathbf{H}}$, a doubly coprime MFD relation on $\tilde{\mathbf{H}}$ can also be given.

Theorem 6.4 *Suppose $P\in\mathbf{M}(\mathbf{G})$, and let $N_pD_p^{-1}$, $\tilde{D}_p^{-1}\tilde{N}_p$ be any right and left coprime MFD of P on $\tilde{\mathbf{H}}$, respectively. Then there exist $U,V,\tilde{U},\tilde{V}\in\mathbf{M}(\tilde{\mathbf{H}})$ such that*

$$\begin{bmatrix}U & V\\ -\tilde{N}_p & \tilde{D}_p\end{bmatrix}\begin{bmatrix}D_p & -\tilde{V}\\ N_p & \tilde{U}\end{bmatrix}=\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} \tag{6.40}$$

Proof. The proof is similar to the 1D case [27, 103]. □

6.4 Output Feedback Practical-Stabilization of nD Systems

In this section, based on the results of previous section, we show that the feedback stabilization problem of nD systems in the practical sense of [1] can be essentially solved by using 1D methods.

Consider the MIMO nD feedback system shown in Figure 6.1 where $N_pD_p^{-1}$, with N_p, $D_p\in\mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$, is a right coprime MFD on $\tilde{\mathbf{H}}$ for the plant $P\in\mathbf{M}(\mathbf{G})$, and $D_c^{-1}[N_{c1}\ N_{c2}]$, with $D_c,N_{c1},N_{c2}\in\mathbf{M}(\tilde{\mathbf{H}})$, is a left coprime MFD on $\tilde{\mathbf{H}}$ for the controller $C\in\mathbf{M}(\mathbf{G})$. Let

$$y=\begin{bmatrix}y_1\\ y_2\end{bmatrix},\quad u=\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} \tag{6.41}$$

Then, the same as shown in Section 4.5, we have

$$y=H_{yu}u \tag{6.42}$$

where,

$$H_{yu}=\begin{bmatrix}D_p\Delta^{-1}N_{c1} & -I+D_p\Delta^{-1}D_c & -D_p\Delta^{-1}N_{c2}\\ N_p\Delta^{-1}N_{c1} & N_p\Delta^{-1}D_c & -N_p\Delta^{-1}N_{c2}\end{bmatrix} \tag{6.43}$$

and

$$\Delta=N_{c2}N_p+D_cD_p. \tag{6.44}$$

If $\det\Delta\neq 0$ and $H_{yu}\in\mathbf{M}(\tilde{\mathbf{H}})$, we say that the nD feedback system of Figure 6.1 is practically-stable. Further, if there exists $C\in\mathbf{M}(\mathbf{G})$ such that $H_{yu}\in\mathbf{M}(\tilde{\mathbf{H}})$, we say that P is practically-stabilizable and that C is a practically-stabilizing compensator for P.

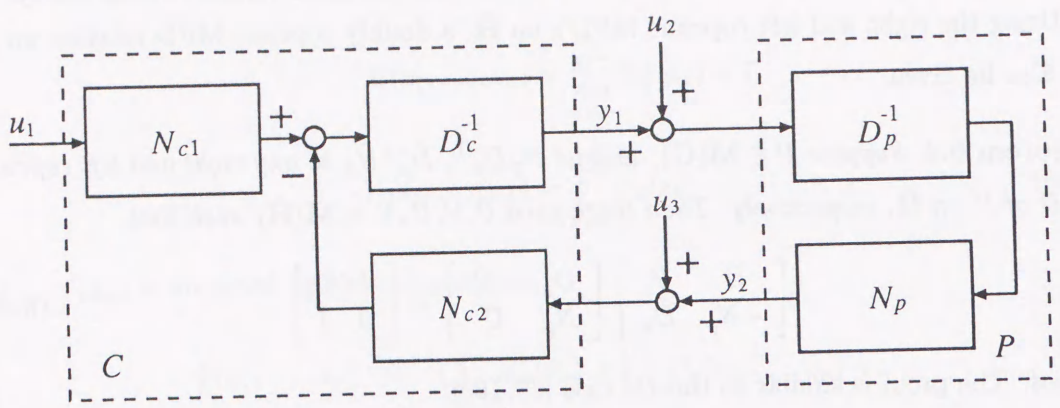


Figure 6.1: nD Feedback Control Systems

Lemma 6.1 *The nD feedback system of Figure 6.1 is practically-stable if and only if Δ is $\tilde{\mathbf{H}}$-unimodular.*

Proof. The sufficiency is obvious. The necessity can be shown in the same way as [103, 104] under the conditions that D_p and N_p are right coprime, D_c, $[N_{c1}\ N_{c2}]$ are left coprime on $\tilde{\mathbf{H}}$. □

Theorem 6.5 *nD plant $P = N_p D_p^{-1} \in \mathbf{M}(\mathbf{G})$ is practically-stabilizable if and only if D_p and N_p are right coprime on $\tilde{\mathbf{H}}$.*

Proof. Suppose D_p and N_p is right coprime on $\tilde{\mathbf{H}}$. Then as shown in the sufficiency proof of Theorem 6.3, the solution $U, V \in \mathbf{M}(\tilde{\mathbf{H}})$ to equation (6.11) can be found. In view of Equations (6.39), (6.34), (6.31) and (6.38), we see that

$$\det U(0,\ldots,0) \neq 0 \tag{6.45}$$

Hence, $C = U^{-1}[N_{c1}\ V] \in \mathbf{M}(\mathbf{G})$ for any $N_{c1} \in \mathbf{M}(\tilde{\mathbf{H}})$. By the fact that I is $\tilde{\mathbf{H}}$-unimodular and in view of Lemma 6.1, the sufficiency is established.

Conversely, if $C = X^{-1}Y = X^{-1}[Y_1\ \ Y_2] \in \mathbf{M}(\mathbf{G})$ is a practically-stabilizing compensator of P, then by Lemma 6.1 we have

$$XD_p + Y_2 N_p = \Phi \tag{6.46}$$

where Φ is $\tilde{\mathbf{H}}$-unimodular. Premultiplying Equation (6.46) by Φ^{-1} gives

$$(\Phi^{-1}X)D_p + (\Phi^{-1}Y_2)N_p = I \tag{6.47}$$

which shows that D_p and N_p is right coprime on $\tilde{\mathbf{H}}$. □

The following theorem gives the parametrization of all nD practically-stabilizing compensators.

Theorem 6.6 *Suppose that $N_p D_p^{-1}$ and $\tilde{D}_p^{-1}\tilde{N}_p$ are respectively any right and left coprime MFD on $\tilde{\mathbf{H}}$ for a given plant $P \in \mathbf{M}(\mathbf{G})$, and that $U, V \in \mathbf{M}(\tilde{\mathbf{H}})$ satisfy $UD_p + VN_p = I$. Then the set of all practically-stabilizing compensators of P is given by*

$$C \in \{(U + S\tilde{N}_p)^{-1}[Q\ \ V - S\tilde{D}_p] \mid$$
$$Q, S \in \mathbf{M}(\tilde{\mathbf{H}}),\ \det(U + S\tilde{N}_p) \in \tilde{\mathbf{I}}\} \tag{6.48}$$

and the set of all possible practically-stable transfer matrices is in the form

$$\begin{bmatrix} D_pQ & D_p(U + S\tilde{N}_p) - I & -D_p(V - S\tilde{D}_p) \\ N_pQ & N_p(U + S\tilde{N}_p) & -N_p(V - S\tilde{D}_p) \end{bmatrix} \tag{6.49}$$

Proof. The sufficiency is obvious. The necessity can be shown in the same way as in Theorem 4.9 by using the results of Lemma 6.1 and Theorem 6.4. □

6.5 Examples

The proposed methods in the previous sections and some properties of practically-stable 2D systems will be illustrated by examples.

Example 6.1. Consider a 2D MIMO system given by the transfer function matrix

$$P(z_1, z_2) = \begin{bmatrix} \dfrac{z_2}{2z_1 - 5} & \dfrac{2z_1 - 1}{6z_1 + 8z_2 - 7} \\ \dfrac{2z_1 - 5}{2z_1 - 1} & \dfrac{z_2}{2z_1 - 1} \end{bmatrix}$$

The right factor coprime MFD of $P(z_1, z_2)$ over $\mathbf{R}[z_1, z_2]$ can be found by the algorithm of [74] or [48] as:

$$P(z_1, z_2) = N_p(z_1, z_2)D_p^{-1}(z_1, z_2)$$

where

$$D_p(z_1,z_2)=\begin{bmatrix}\frac{1}{16}(2z_1-1)(2z_1-5) & -\frac{1}{2}(2z_2-1)(2z_1-5)z_2\\ 0 & 2(6z_1+8z_2-7)\end{bmatrix}$$

$$N_p(z_1,z_2)=\begin{bmatrix}\frac{1}{16}(2z_1-1)z_2 & \frac{1}{2}(8z_1-2z_2^3+z_2^2-4)\\ \frac{1}{16}(2z_1-5)^2 & -\frac{1}{2}(4z_1z_2-2z_1-18z_2-3)z_2\end{bmatrix}$$

By making use of the results of Section 4.2 (see also [114]), it is easy to verify that $(z_1,z_2)=(1/2,1/2)\in\bar{\mathbb{U}}^2$ is a common zero of all the 2×2 minors of the matrix $[D_p^T(z_1,z_2)\quad N_p^T(z_1,z_2)]^T$. Therefore, P is unstabilizable in the sense of conventional stability (see Chapter 4 or, e.g., [49, 114]).

However, the right greatest common factor $G(z_1)$ of

$$D_p(z_1,0)=\begin{bmatrix}\frac{1}{16}(2z_1-1)(2z_1-5) & 0\\ 0 & 2(6z_1-7)\end{bmatrix},$$

$$N_p(z_1,0)=\begin{bmatrix}0 & 2(2z_1-1)\\ \frac{1}{16}(2z_1-5)^2 & 0\end{bmatrix}$$

is

$$G(z_1)=\begin{bmatrix}\frac{1}{2}(2z_1-5) & 0\\ 0 & 1\end{bmatrix},$$

and it is obvious that $\mathcal{V}(\mathcal{I}_1)\cap\bar{\mathbb{U}}=\{5/2\}\cap\bar{\mathbb{U}}=\emptyset$. In addition, we can easily verify that the matrices

$$D_p(0,z_2)=\begin{bmatrix}\frac{5}{16} & \frac{5}{2}(2z_2-1)z_2\\ 0 & 2(8z_2-7)\end{bmatrix},$$

$$N_p(0,z_2)=\begin{bmatrix}-\frac{z_2}{16} & -\frac{1}{2}(2z_2^3-z_2^2+4)\\ \frac{25}{16} & \frac{3}{2}z_2(6z_2+1)\end{bmatrix}$$

are right coprime over $\mathbf{R}[z_2]$, i.e., $\mathcal{V}(\mathcal{I}_2)=\emptyset$. In view of Theorems 6.5 and 6.3, P is practically-stabilizable.

The practically-stabilizing compensator for P can be constructed by the following

procedure. First, we find the solutions to Equation (6.16).

$$\tilde{X}_1(z_1)=\begin{bmatrix}2 & 0\\ 0 & -\frac{1}{8}\end{bmatrix}\qquad \tilde{Y}_1(z_1)=\begin{bmatrix}0 & -2\\ \frac{3}{8} & 0\end{bmatrix}$$

$$\tilde{\Phi}_1(z_1)=\begin{bmatrix}\frac{1}{2}(2z_1-5) & 0\\ 0 & 1\end{bmatrix}$$

$$\tilde{X}_2(z_2)=\begin{bmatrix}\frac{4}{5}(2z_2^3-z_2^2+4) & 0\\ -\frac{z_2}{10} & 0\end{bmatrix}\qquad \tilde{Y}_2(z_2)=\begin{bmatrix}4z_2(2z_2-1) & 0\\ -\frac{1}{2} & 0\end{bmatrix}$$

$$\tilde{\Phi}_2(z_2)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

where, obviously, $\det\tilde{\Phi}_1(z)=(2z-5)/2\neq 0$, $\det\tilde{\Phi}_2(z)=1\neq 0$, $\forall z\in\bar{\mathbb{U}}$.

Then, $\tilde{D}_1,\tilde{N}_1$ & $\tilde{D}_2,\tilde{N}_2$ that satisfy Equation (6.19) can be obtained.

$$\tilde{D}_1(z_1)=\begin{bmatrix}\frac{3}{4}(6z_1-7) & 0\\ 0 & -\frac{1}{4}(2z_1-1)\end{bmatrix}$$

$$\tilde{N}_1(z_1)=\begin{bmatrix}0 & \frac{3}{4}(2z_1-1)\\ -\frac{1}{4}(2z_1-5) & 0\end{bmatrix}$$

$$\tilde{D}_2(z_2)=\begin{bmatrix}-(8z_2-7) & 0\\ (8z_2-7)z_2 & -1\end{bmatrix}$$

$$\tilde{N}_2(z_2)=\begin{bmatrix}\frac{1}{5}(8z_2-7)z_2 & 1\\ -\frac{1}{5}(8z_2^3-7z_2^2+25) & 0\end{bmatrix}$$

Since $\det\tilde{D}_1(0)=-21/16\neq 0$, $\det\tilde{D}_2(0)=-7\neq 0$, we can set R_k, $k=1,2$ in Equation (6.22) as

$$R_1=\tilde{Y}_1(0)\tilde{D}_1^{-1}(0)=\begin{bmatrix}0 & -8\\ -\frac{1}{14} & 0\end{bmatrix},$$

$$R_2=\tilde{Y}_2(0)\tilde{D}_2^{-1}(0)=\begin{bmatrix}0 & 0\\ -\frac{1}{14} & 0\end{bmatrix}.$$

Substitution of these results to Equation (6.18) gives

$$\bar{X}_1(z_1) = \tilde{X}_1(z_1) + R_1\tilde{N}_1(z_1) = \begin{bmatrix} 4(z_1-2) & 0 \\ 0 & -\frac{1}{28}(3z_1+2) \end{bmatrix}$$

$$\bar{Y}_1(z_1) = \tilde{Y}_1(z_1) - R_1\tilde{D}_1(z_1) = \begin{bmatrix} 0 & -4z_1 \\ \frac{9}{28}z_1 & 0 \end{bmatrix}$$

$$\bar{X}_2(z_2) = \tilde{X}_2(z_2) + R_2\tilde{N}_2(z_2) = \begin{bmatrix} \frac{4}{5}(2z_2^3 - z_2^2 + 4) & 0 \\ -\frac{4}{35}z_2^2 & -\frac{1}{14} \end{bmatrix}$$

$$\bar{Y}_2(z_2) = \tilde{Y}_2(z_2) - R_2\tilde{D}_2(z_2) = \begin{bmatrix} 4z_2(2z_2-1) & 0 \\ -\frac{4}{7}z_2 & 0 \end{bmatrix}$$

The facts that $\det \bar{X}_1(0) = 4/7 \neq 0$, $\det \bar{X}_2(0) = -8/35 \neq 0$ imply the existence of $\bar{X}_1^{-1}(0)$ and $\bar{X}_2^{-1}(0)$. Hence, Equation (6.27) is solvable and it admits the solution:

$$X_1(z_1) = \bar{X}_1^{-1}(0)\bar{X}_1(z_1) = \begin{bmatrix} \frac{1}{2}(2-z_1) & 0 \\ 0 & \frac{1}{2}(3z_1+2) \end{bmatrix}$$

$$Y_1(z_1) = \bar{X}_1^{-1}(0)\bar{Y}_1(z_1) = \begin{bmatrix} 0 & \frac{1}{2}z_1 \\ -\frac{9}{2}z_1 & 0 \end{bmatrix}$$

$$\Phi_1(z_1) = \bar{X}_1^{-1}(0)\bar{\Phi}_1(z_1) = \begin{bmatrix} -\frac{1}{16}(2z_1-5) & 0 \\ 0 & -14 \end{bmatrix}$$

$$X_2(z_2) = \bar{X}_2^{-1}(0)\bar{X}_2(z_2) = \begin{bmatrix} \frac{1}{4}(2z_2^3 - z_2^2 + 4) & 0 \\ \frac{8}{5}z_2^2 & 1 \end{bmatrix}$$

$$Y_2(z_2) = \bar{X}_2^{-1}(0)\bar{Y}_2(z_2) = \begin{bmatrix} \frac{5}{4}z_2(2z_2-1) & 0 \\ 8z_2 & 0 \end{bmatrix}$$

$$\Phi_2(z_2) = \bar{X}_2^{-1}(0)\bar{\Phi}_2(z_2) = \begin{bmatrix} \frac{5}{16} & 0 \\ 0 & -14 \end{bmatrix}$$

By the results of $X_1(0) = I$, $X_2(0) = I$ and Equation (6.28), we get

$$X_1'(z_1) = X_1(z_1) - I = \begin{bmatrix} -\frac{1}{2}z_1 & 0 \\ 0 & \frac{3}{2}z_1 \end{bmatrix}$$

$$X_2'(z_2) = X_2(z_2) - I = \begin{bmatrix} \frac{1}{4}(2z_2-1)z_2^2 & 0 \\ \frac{8}{5}z_2^2 & 0 \end{bmatrix}$$

with $X_1'(0) = 0$, $X_2'(0) = 0$.

Now, by Equations (6.34) and (6.35), the solution $X(z_1,z_2)$, $Y(z_1,z_2)$ to Equation (6.14) is constructed as

$$X(z_1,z_2) = X_1'(z_1) + X_2'(z_2) + I$$

$$= \begin{bmatrix} -\frac{1}{4}(2z_1 - 2z_2^3 + z_2^2 - 4) & 0 \\ \frac{8}{5}z_2^2 & \frac{1}{2}(3z_1+2) \end{bmatrix}$$

$$Y(z_1,z_2) = Y_1(z_1) + Y_2(z_2) = \begin{bmatrix} \frac{5}{4}z_2(2z_2-1) & \frac{1}{2}z_1 \\ -\frac{1}{2}(9z_1 - 16z_2) & 0 \end{bmatrix}$$

Moreover, in light of Equation (6.36), $\Phi(z_1,z_2)$ is computed as follows.

$$\Phi(z_1,z_2) = \begin{bmatrix} \Phi_{11}(z_1,z_2) & \Phi_{12}(z_1,z_2) \\ \Phi_{21}(z_1,z_2) & \Phi_{22}(z_1,z_2) \end{bmatrix}$$

where

$$\Phi_{11}(z_1,z_2) = \frac{4z_2^3z_1^2 - 2z_2^3z_1 - 2z_2^2z_1^2 + z_2^2z_1 - 4z_1 + 10}{32}$$

$$\Phi_{12}(z_1,z_2) = -\frac{z_1z_2(4z_2^4 - 4z_2^3 + z_2^2 - 40z_2 + 8)}{4}$$

$$\Phi_{21}(z_1,z_2) = \frac{z_1z_2(2z_1-1)(32z_2-45)}{160}$$

$$\Phi_{22}(z_1,z_2) = -\frac{z_1z_2(64z_2^3 - 122z_2^2 + 45z_2 - 1120) + 280}{20}$$

and it is trivial to verify that $\Phi(z_1,0) = \Phi_1(z_1)$ and $\Phi(0,z_2) = \Phi_2(z_2)$.

Finally, we can find $U(z_1,z_2)$, $V(z_1,z_2)$ by Equation (6.39), and construct the class of all practically-stabilizing compensators of P according to Theorem 6.6.

Example 6.2 Consider continuously the plant and the designed closed feedback system of Example 6.1. If we set Q, the free parameter in the closed-loop transfer matrix of Equation (6.49), as

$$Q(z_1, z_2) = q(z_1, z_2)^{-1} I \in \mathbf{M}(\tilde{\mathbf{H}}) \tag{6.50}$$

where

$$q(z_1, z_2) = 3 + z_1 + z_2 - 1.08 z_1 z_2,$$

we have the transfer matrix from the input u_1 to the output y_2

$$H_{y_2 u_1} = N_p Q = q^{-1} N_p \triangleq \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}.$$

Since for brevity we will only show the simulation result for h_{12}, we just give h_{12} explicitly here, i.e.,

$$h_{12}(z_1, z_2) = \frac{8z_1 - 2z_2^3 + z_2^2 - 4}{2(3 + z_1 + z_2 - 1.08 z_1 z_2)}$$

By the methods shown in Section 3.5, one can confirm that $q(z_1, z_2)$ has zero in $\bar{U}^2$, i.e., it is not BIBO stable. By Theorem 6.2, however, it is easy to verify that $q(z_1, z_2)$ is practically-BIBO stable.

Figure 6.2 shows the simulation of the impulse response of $h_{12}(z_1, z_2)$. From the result of Figure 6.2, we can see a character commonly for practically-BIBO stable impulse responses, namely, if osillation occurs in such a response it must appear approximately along the disgonal of the cooordinate plain. In fact, this is quite reasonable because that, in view of the definition of practical-BIBO stability, any practically-BIBO stable (2D) impulse response has to obey the following properties: if i_1 is restricted to a fixed finite value while i_2 is free, then the response must converge to zero as i_2 increases largely enough; and the roles of i_1 and i_2 can be interchanged; but the response is not necessary to converge to zero when i_1 and i_2 approach to infinity simultaneously. It should be apparent that these properties are not contradictory to the fact that a practiclly-BIBO stable system may be unstable from the viewpoint of conventional-BIBO stability.

Using the same denominator $q(z_1, z_2)$ but the different numerator

$$b(z_1, z_2) = 1 + 2z_1 + z_1^2 + 2z_2 + z_2^2 + 4z_1 z_2 + 2z_1 z_2^2 + 2z_1^2 z_2 + z_1^2 z_2^2,$$

we get another system $b(z_1, z_2)/q(z_1, z_2)$, which has the impulse response shown in Figure

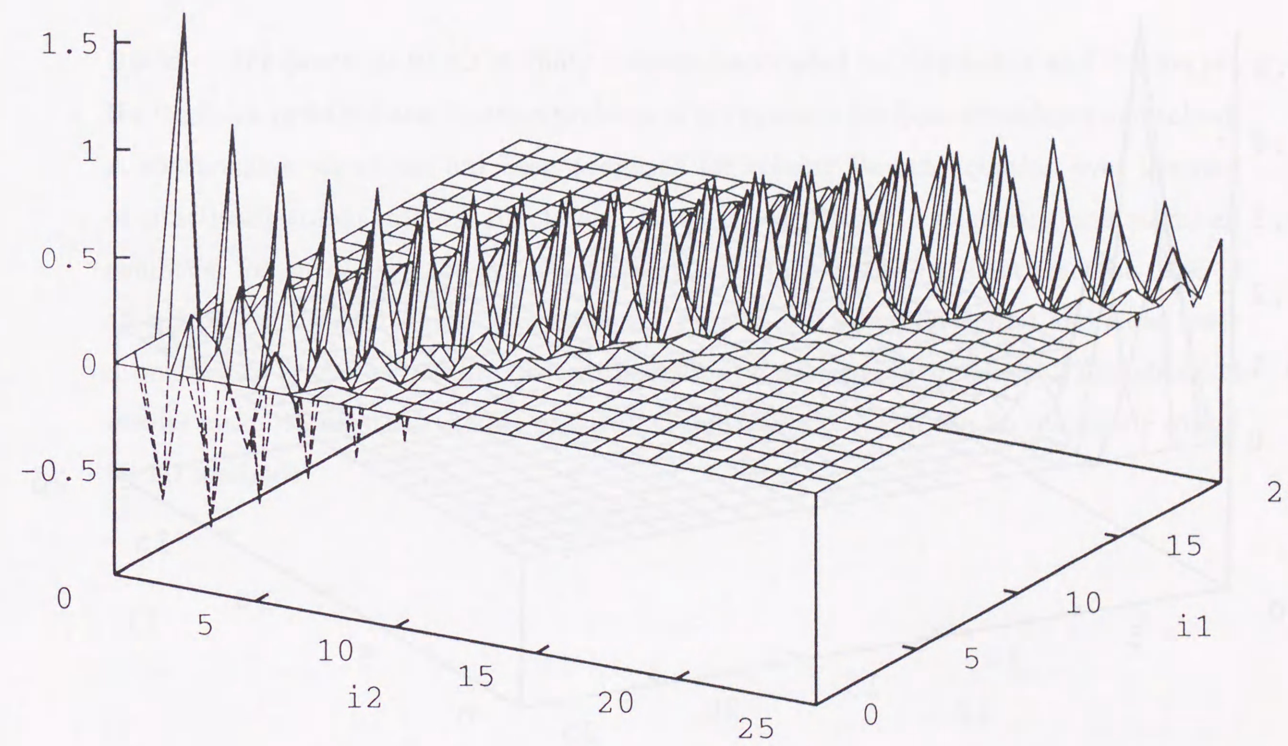


Figure 6.2: Impulse response of $h_{12}(z_1, z_2)$

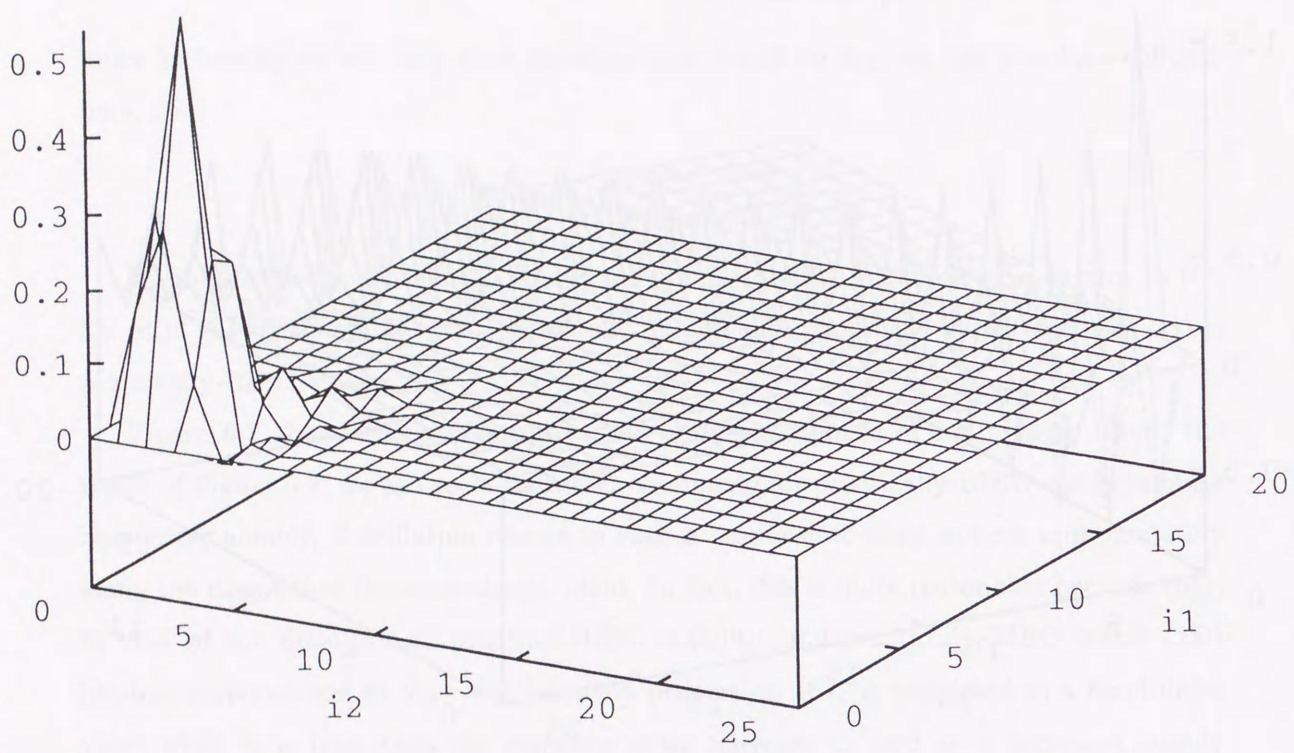


Figure 6.3: Impulse response of $b(z_1, z_2)/q(z_1, z_2)$

6.3. By this result, we confirmed the existence of practically-BIBO stable system with rather satisfactory dynamical property even though it does not satisfy the conventional-BIBO stability. This can also be viewed as an example that proves the conclusion of [45]: there may exist 2D systems which is BIBO unstable but has a summable impulse response. Further, comparing the results of Figures 6.2 and 6.3, it is easy to see that zeros of the transfer function influence the dynamical property of the corresponding system greatly, which implies that to design satisfactory practically-stable feedback systems the influence of zeros should also be investigated and considered carefully.

6.6 Summary

Based on the practical-BIBO stability concept introduced by Agathoklis and Bruton [1], the feedback practical-stabilization problem of nD systems has been considered and solved. A constructive algorithm has been proposed for solving Bezout equation over the ring of practically-stable rational functions. Further, we derived a necessary and sufficient condition for an nD system to be practically-stabilizable, and parametrized the class of all nD practically-stabilizing compensators. Finally, the proposed methd and some basic properties of practically-stable systems have been illustrated by examples. The obtained results make it clear that the nD practical-stabilization problem can be essentially solved by 1D methods.

Chapter 7

Practical-Stabilization of nD Systems by State-Space Approach

7.1 Introduction

The purpose of this chapter is to consider some fundamental problems of nD discrete systems in the practical sense of [1] from state-space point of view. In Section 7.2, based on nD Roesser state-space model, we give the definition of *practical internal stability* or *practical asymptotic stability* and show a necessary and sufficient condition for the defined stability. In Section 7.3, concepts of practical-controllability and practical-observability are introduced and associated conditions are derived. Based on these results, in Section 7.4, we define practical-stabilizability and practical-detectability, and meanwhile solve the problem of practical-stabilization by local state feedback and the problem of construction of state observer in the practical sense. Section 7.5 makes it clear that the practical-BIBO stability coincides with practical internal stability under the concepts of practical-stabilizability and practical-detectability. Finally, a connection between the algebra and state-space approaches will be clarified in Section 7.6.

7.2 Practical Internal Stability for nD Discrete Systems

In this section, the internal stability, or asymptotic stability of nD systems whose input and output signals are unbounded in, at most, one dimension will be considered. Based on the nD Roesser state-space model (3.30), we introduce the concept of practical internal stability and derive a necessary and sufficient condition for it.

For convenience, the nD Roesser state-space model (3.30) is rewritten here, i.e.,

$$\begin{cases} x'(i_1,\ldots,i_n) = Ax(i_1,\ldots,i_n) + Bu(i_1,\ldots,i_n) \\ y(i_1,\ldots,i_n) = Cx(i_1,\ldots,i_n) + Du(i_1,\ldots,i_n) \end{cases} \tag{7.1}$$

where $u(i_1,\ldots,i_n) \in \mathbf{R}^m$ and $y(i_1,\ldots,i_n) \in \mathbf{R}^l$ are the input and output vectors, respectively; $x(i_1,\ldots,i_n) \in \mathbf{R}^{\tilde{n}}$ is the local state vector with $x^T = [x_1^T \cdots x_n^T]^T$, $x_i \in \mathbf{R}^{n_i}$, $i = 1,\ldots,n$, $\tilde{n} = \sum_{i=1}^n n_i$; A, B, C and D are constant matrices of suitable dimensions (see Section 3.4 for detail).

As in [15], we define an alternative nD z-transformation:

$$\begin{aligned} f(z_1,\ldots,z_n) &= \mathcal{Z}(f(i_1,\ldots,i_n)) \\ &= \sum_{i_1+\ldots+i_n \geq 0} f(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n}. \end{aligned} \tag{7.2}$$

Applying this z-transformation to nD system (7.1) gives

$$x(z_1,\ldots,z_n) = (I - ZA)^{-1} ZBu(z_1,\ldots,z_n) + (I - ZA)^{-1}\mathcal{X}_0 \tag{7.3}$$

$$y(z_1,\ldots,z_n) = Cx(z_1,\ldots,z_n) + Du(z_1,\ldots,z_n) \tag{7.4}$$

where

$$Z = \begin{bmatrix} I_{m_1}z_1 & 0 & \cdots & 0 \\ 0 & I_{m_2}z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_n}z_n \end{bmatrix} \tag{7.5}$$

and

$$\mathcal{X}_0 = \sum_{i_1+\ldots+i_n=0} x(i_1,\ldots,i_n) z_1^{i_1} \cdots z_n^{i_n} \tag{7.6}$$

Assuming zero initial condition $\mathcal{X}_0 = 0$, the transfer matrix

$$G(z_1,\ldots,z_n) = C(I - ZA)^{-1} ZB + D \tag{7.7}$$

gives the input/output relation

$$y(z_1,\ldots,z_n) = G(z_1,\ldots,z_n) u(z_1,\ldots,z_n). \tag{7.8}$$

The characteristic polynomial of nD system (3.30) [5] is defined as

$$\rho(z_1,\ldots,z_n) = \det(I - ZA) \tag{7.9}$$

and the following lemma can be given.

Lemma 7.1 *The following relation holds for the characteristic polynomial* (3.43) *of system* (7.1).

$$\rho(0,\ldots,z_k,\ldots,0) = \det(I_{n_k} - z_k A_{kk}), \quad k = 1,\ldots,n \tag{7.10}$$

Proof. Substituting $z_i = 0$, $i = 1,\ldots,n$, $i \neq k$, into Equation (3.43) gives

$$\rho(0,\ldots,z_k,\ldots,0) = \det \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & I_{n_2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ -z_kA_{k1} & -z_kA_{k2} & \cdots & I_{n_k} - z_kA_{kk} & \cdots & -z_kA_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_{n_n} \end{bmatrix}. \tag{7.11}$$

Now, applying Laplace expansion to Equation (7.11) the proof can be shown. □

Now, introduce the following notation for *global* state $\tilde{\mathcal{X}}_r$ in the practical sense mentioned above.

$$\tilde{\mathcal{X}}_r = \{x(i_1,\ldots,i_n) \in \mathbf{R}^{\tilde{n}} \mid \sum_{j=1}^n i_j = r,\ (i_1,\ldots,i_n) \in \mathbf{Z}_+^{-n}\} \tag{7.12}$$

Denote by $\|x(i_1,\ldots,i_n)\|$ the Euclidean norm of *local* state $x(i_1,\ldots,i_n)$ in the state space $\mathbf{R}^{\tilde{n}}$ and define the following norm for $\tilde{\mathcal{X}}_r$.

$$\|\tilde{\mathcal{X}}_r\| = \max_{h \leq h_0}\{\|x(i_1,\ldots,(r-h)_j,\ldots,i_n)\| \mid h = \sum_{\substack{k=1 \\ k\neq j}}^n i_k,\ j = 1,\ldots,n\} \tag{7.13}$$

where h_0 is any finite integer.

Definition 7.1 *For nD system* (7.1)*, suppose that $u = 0$ and $\|\tilde{\mathcal{X}}_0\|$ is finite. Then, the system* (7.1) *is practically asymptotically stable if $\|\tilde{\mathcal{X}}_r\| \to 0$ as $r \to \infty$.*

Theorem 7.1 *nD system* (7.1) *is practically asymptotically stable if and only if all the matrices $A_{kk}, k = 1,\ldots,n$ are stable in the* (1D) *sense, namely,*

$$\rho(0,\ldots,z_k,\ldots,0) = \det(I_{m_k} - z_kA_{kk}) \neq 0, \quad \forall z_k \in \bar{U},\ k = 1,\ldots,n. \tag{7.14}$$

The theorem can be proved by combining the ideas adopted in [1, 40, 41, 46] as follows.

Proof. Sufficiency: From Equation (7.3), we get, for $u(i_1, \ldots, i_n) = 0$,

$$x(z_1, \ldots, z_n) = (I - ZA)^{-1}\mathcal{X}_0 \triangleq \frac{R(z_1, \ldots, z_n)}{\rho(z_1, \ldots, z_n)}\mathcal{X}_0 \tag{7.15}$$

where

$$R(z_1, \ldots, z_n) = \mathrm{adj}(I - ZA) \in \mathbf{M}(\mathbf{R}[z_1, \ldots, z_n]), \tag{7.16}$$

$$\rho(z_1, \ldots, z_n) = \det(I - ZA) \in \mathbf{R}[z_1, \ldots, z_n]. \tag{7.17}$$

Since

$$\rho(0, \ldots, 0) = \det(I - ZA)|_{z_1 = \cdots = z_n = 0} = 1 \neq 0, \tag{7.18}$$

there is some $\epsilon > 0$ such that $(I - ZA)^{-1}$ is analytic in the region

$$U_\epsilon^n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_i| < \epsilon,\ i = 1, \ldots, n\} \tag{7.19}$$

and has the series expansion in U_ϵ^n

$$\frac{R(z_1, \ldots, z_n)}{\rho(z_1, \ldots, z_n)} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} M_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}. \tag{7.20}$$

For $(z_1, \ldots, z_n) \in U_\epsilon^n$, Equation (7.20) can be rewritten as

$$\frac{R(z_1, \ldots, z_n)}{\rho(z_1, \ldots, z_n)} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} Q_{i_1, \ldots, i_j, \ldots, i_n}(z_k) z_1^{i_1} \cdots z_j^{i_j} \cdots z_n^{i_n},$$
$$j = 1, \ldots, n, \quad j \neq k \tag{7.21}$$

with

$$Q_{i_1, \ldots, i_j, \ldots, i_n}(z_k) = \sum_{i_k=0}^{\infty} M_{i_1, \ldots, i_k, \ldots, i_n} z_k^{i_k}, \quad i_j = i_1, \ldots, i_n, \quad i_j \neq i_k \tag{7.22}$$

Equation (7.21) can be differentiated termwise for $(z_1, \ldots, z_n) \in U_\epsilon^n$, and this gives the result

$$\frac{\partial^{(i_1 + \cdots + i_j + \cdots + i_n)}}{\partial z_1^{i_1}, \ldots, \partial z_j^{i_j}, \ldots, \partial z_n^{i_n}} \left[\frac{R(z_1, \ldots, z_n)}{\rho(z_1, \ldots, z_n)}\right]_{z_1 = 0, \ldots, z_j = 0, \ldots, z_n = 0}$$
$$= (i_1!) \cdots (i_j!) \cdots (i_n!) Q_{i_1, \ldots, i_j, \ldots, i_n}(z_k), \quad j = 1, \ldots, n, \quad j \neq k. \tag{7.23}$$

The left-hand side of Equation (7.23) is a 1D rational function of the form

$$\frac{\tilde{R}_{i_1, \ldots, i_j, \ldots, i_n}(z_k)}{[\rho(0, \ldots, z_k, \ldots, 0)]^{(i_1+1) + \cdots + (i_j+1) + \cdots + (i_n+1)}}$$
$$j = 1, \ldots, n, \qquad j \neq k. \tag{7.24}$$

Therefore, if

$$\rho(0, \ldots, z_k, \ldots, 0) = \det(I_{m_k} - z_k A_{kk}) \neq 0 \qquad \text{for } z_k \in \bar{U} \tag{7.25}$$

then for all finite i_j, $j = 1, \ldots, n$, $j \neq k$

$$\sum_{i_k=0}^{\infty} \|M_{i_1, \ldots, i_k, \ldots, i_n}\| < \infty, \quad k = 1, \ldots, n. \tag{7.26}$$

As a sum of finite number of the form of Equation (7.26), then, the following relation holds true for any finite $N_1, N_2, \ldots, N_n$.

$$\sum_{i_1=0}^{N_1} \cdots \sum_{i_k=0}^{\infty} \cdots \sum_{i_n=0}^{N_n} \|M_{i_1, \ldots, i_k, \ldots, i_n}\| < \infty, \quad k = 1, \ldots, n \tag{7.27}$$

Since every absolutely convergent series can be rearranged freely without affecting convergence of the sum, Equation (7.27) can be reexpressed in the following form (for 2D case, see the example of Figure 7.1).

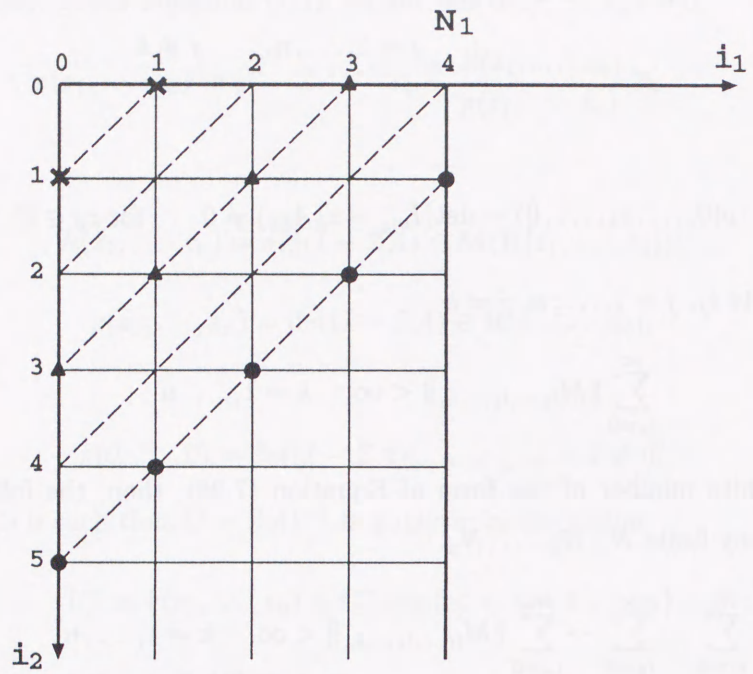
$$\sum_{i_1=0}^{N_1} \cdots \sum_{i_k=0}^{\infty} \cdots \sum_{i_n=0}^{N_n} \|M_{i_1, \ldots, i_k, \ldots, i_n}\| = \sum_{r_k=0}^{\infty} \Bigg\{ \sum_{\substack{h_k \le \min\{h_0, r_k\} \\ i_j \le N_j, j \neq k}} \|M_{i_1, \ldots, r_k - h_k, \ldots, i_n}\| \Bigg\}$$
$$\triangleq \sum_{r_k=0}^{\infty} \|M_{r_k}\| < \infty \tag{7.28}$$

where

$$r_k = \sum_{j=1}^{n} i_j, \qquad h_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} i_j, \qquad h_0 = \sum_{\substack{j=1 \\ j \neq k}}^{n} N_j < \infty, \qquad k = 1, \ldots, n.$$

Then, it turns out to be clear that

$$\lim_{r_k \to \infty} \|M_{r_k}\| = \lim_{r_k \to \infty} \Bigg\{ \sum_{\substack{h_k \le \min\{h_0, r_k\} \\ i_j \le N_j, j \neq k}} \|M_{i_1, \ldots, r_k - h_k, \ldots, i_n}\| \Bigg\} = 0 \tag{7.29}$$



$$\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{\infty}\|M_{i_1 i_2}\| \quad \Leftrightarrow \quad \sum_{r_2=0}^{\infty}\sum_{h_2\le\min\{N_1,r_2\}}\|M_{i_1,r_2-h_2}\|$$

$$i_1 = r_2 - i_2, \quad i_2 = r_2 - h_2$$

$\triangle$	$r_2 = 3:$	$h_2 = 0$	$i_1 = 0$	$i_2 = 3$
		$h_2 = 1$	$i_1 = 1$	$i_2 = 2$
		$h_2 = 2$	$i_1 = 2$	$i_2 = 1$
		$h_2 = 3$	$i_1 = 3$	$i_2 = 0$
		$\vdots$		
$\bullet$	$r_2 = 5:$	$h_2 = 0$	$i_1 = 0$	$i_2 = 5$
		$h_2 = 1$	$i_1 = 1$	$i_2 = 4$
		$h_2 = 2$	$i_1 = 2$	$i_2 = 3$
		$h_2 = 3$	$i_1 = 3$	$i_2 = 2$
		$h_2 = 4$	$i_1 = 4$	$i_2 = 1$

Figure 7.1: Example for the case of $n = 2$

On the other hand, Equation (7.20) can be also equivalently written as

$$\frac{R(z_1,\ldots,z_n)}{\rho(z_1,\ldots,z_n)} = \sum_{i_1+\cdots+i_n\ge 0} M_{i_1,\ldots,i_n} z_1^{i_1}\cdots z_n^{i_n}. \tag{7.30}$$

Then, from Equations (7.15) and (7.30), we have that for finite $\|\tilde{\mathcal{X}}_0\|$ and any local state

$$\boldsymbol{x}(l_1,\ldots,r_k-h_k,\ldots,l_n), \quad h_k = \sum_{\substack{j=1\\ j\neq k}}^{n} l_j < \infty, \quad r_k = r, \quad k = 1,\ldots,n$$

in $\tilde{\mathcal{X}}_r$ with $r > 0$, the following relation holds.

$$\begin{aligned}
&\|\boldsymbol{x}(l_1,\ldots,r_k-h_k,\ldots,l_n)\| = \\
&= \Big\| \sum_{\substack{i_1+\cdots+i_n=r_k\\ (i_1,\ldots,i_n)\in Z_{+k}^{-n}}} M_{i_1,\ldots,i_n}\boldsymbol{x}(l_1-i_1,\ldots,r_k-h_k-i_k,\ldots,l_n-i_n)\Big\| \\
&\le \sum_{\substack{i_1+\cdots+i_n=r_k\\ (i_1,\ldots,i_n)\in Z_{+k}^{-n}}} \|M_{i_1,\ldots,i_n}\|\|\boldsymbol{x}(l_1-i_1,\ldots,r_k-h_k-i_k,\ldots,l_n-i_n)\| \\
&\le \|\tilde{\mathcal{X}}_0\| \sum_{\substack{i_1+\cdots+i_n=r_k\\ (i_1,\ldots,i_n)\in Z_{+k}^{-n}}} \|M_{i_1,\ldots,i_n}\|, \qquad k = 1,\ldots,n
\end{aligned} \tag{7.31}$$

According to Equation (7.29) and Definition 7.1, this implies the practical asymptotic stability of nD system (7.1).

Necessity: We can express any local state

$$\boldsymbol{x}(l_1,\ldots,r_k-h_k,\ldots,l_n), \quad h_k = \sum_{\substack{j=1\\ j\neq k}}^{n} l_j < \infty, \quad r_k = r, \quad k = 1,\ldots,n$$

in $\tilde{\mathcal{X}}_r$ as

$$x^q(l_1,\ldots,r_k-h_k,\ldots,l_n) = \sum_{\substack{i_1+\cdots+i_n=r_k\\ (i_1,\ldots,i_n)\in Z_{+k}^{-n}}} M^q_{i_1,\ldots,i_n}\boldsymbol{x}(l_1-i_1,\ldots,r_k-h_k-i_k,\ldots,l_n-i_n)$$

$$q = 1,\ldots,\tilde{n} \tag{7.32}$$

where x^q is the qth element of $x \in \mathbf{R}^{\tilde{n}}$, and $M^q_{i_1,\ldots,i_n}$ is the qth row of $M_{i_1,\ldots,i_n} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$.

Suppose that nD system (7.1) is practically asymptotically stable. By definition 7.1, then, for any $\|\tilde{\mathcal{X}}_0\| < \infty$ we have

$$\lim_{r_k \to \infty} \|\boldsymbol{x}(l_1, \ldots, r_k - h_k, \ldots, l_n)\| = 0. \tag{7.33}$$

Using the equivalenc of norms [28, 60], it can be shown that this result is equivalent to

$$\lim_{r_k \to \infty} |x^q(l_1, \ldots, r_k - h_k, \ldots, l_n)| = 0, \quad q = 1, \ldots, \tilde{n}. \tag{7.34}$$

Let

$$\boldsymbol{x}(l_1 - i_1, \ldots, r_k - h_k - i_k, \ldots, l_n - i_n) = \operatorname{sgn}\left(M^q_{i_1, \ldots, i_n}\right)^T \tag{7.35}$$

It follows then from Equation (7.32) that

$$\begin{aligned} |x^q(l_1, \ldots, r_k - h_k, \ldots, l_n)| &= \sum_{\substack{i_1 + \cdots + i_n = r_k \\ (i_1, \ldots, i_n) \in Z^{-n}_{+k}}} |M^q_{i_1, \ldots, i_n}| \\ &= |M^q_{0, \ldots, r_k, \ldots, 0}| + \sum_{\substack{i_1 + \cdots + i_n = r_k, (i_1, \ldots, i_n) \in Z^{-n}_{+k} \\ (i_1, \ldots, i_k, \ldots, i_n) \neq (0, \ldots, r_k, \ldots, 0)}} |M^q_{i_1, \ldots, i_n}| \\ &\geq |M^q_{0, \ldots, r_k, \ldots, 0}|. \end{aligned} \tag{7.36}$$

Therefore, in view of Equation (7.34),

$$\lim_{i_k \to \infty} |M^q_{0, \ldots, i_k, \ldots, 0}| = 0. \tag{7.37}$$

which shows that for $z_k \in$U

$$\sum_{i_k = 0}^{\infty} M^q_{0, \ldots, i_k, \ldots, 0} z_k^{i_k} < \infty. \tag{7.38}$$

On the other hand, assuming $z_i = 0$, $i = 1, \ldots, n, i \neq k$ in Equation (7.20), we get

$$\frac{R(0, \ldots, z_k, \ldots, 0)}{\rho(0, \ldots, z_k, \ldots, 0)} = \sum_{i_k = 0}^{\infty} M_{0, \ldots, i_k, \ldots, 0} z_k^{i_k}. \tag{7.39}$$

Let $R^q(0, \ldots, z_k, \ldots, 0)$ denote the qth row of $R(0, \ldots, z_k, \ldots, 0)$. Then it follows that

$$\frac{R^q(0, \ldots, z_k, \ldots, 0)}{\rho(0, \ldots, z_k, \ldots, 0)} = \sum_{i_k = 0}^{\infty} M^q_{0, \ldots, i_k, \ldots, 0} z_k^{i_k}. \tag{7.40}$$

Since the left-hand side of Equation (7.40) is 1D rational function, it is easy to see that the conclusion of Equation (7.38) in fact means that for any $z_k \in$ U

$$\rho(0, \ldots, z_k, \ldots, 0) \neq 0. \tag{7.41}$$

To complete the proof, we next show that $\rho(0, \ldots, z_k, \ldots, 0) \neq 0$ for any $z_k \in$T. Assume that $\boldsymbol{u}(i_1, \ldots, i_n) = 0$ and the boundary condition $\boldsymbol{x}_j(i_1, \ldots, 0_j, \ldots, i_n) = 0$, $j = 1, \ldots, n$ except $\boldsymbol{x}(0, \ldots, 0) \neq 0$. From Equation (3.35), then, the following relation can be obtained.

$$\boldsymbol{x}(0, \ldots, i_k, \ldots, 0) = A^{0, \ldots, i_k, \ldots, 0} \boldsymbol{x}(0, \ldots, 0), \qquad k = 1, \ldots, n. \tag{7.42}$$

By simple calculation, we see that

$$\boldsymbol{x}_j(0, \ldots, i_k, \ldots, 0) = \begin{cases} 0, & j \neq k \\ A_{kk}^{i_k - 1} \sum_{i=1}^{n} A_{ki} \boldsymbol{x}_k(0, \ldots, 0) & j = k \end{cases} \tag{7.43}$$

Let $a_k \in$T and

$$\rho(0, \ldots, a_k, \ldots, 0) = \det(I - A_{kk} a_k) = 0. \tag{7.44}$$

Then, there exists a nonzero vector $\nu \in \mathbf{R}^{n_k}$ that satisfies

$$A_{kk} a_k \nu = \nu. \tag{7.45}$$

Let $\boldsymbol{x}_i(0, \ldots, 0) = 0$, $i = 1, \ldots, n$, $i \neq k$, and

$$\boldsymbol{x}_k(0, \ldots, 0) = \alpha\nu + \bar{\alpha}\bar{\nu} \tag{7.46}$$

with

$$\alpha = e^{j\psi} \in \mathrm{T}, \quad a_k = e^{-j\phi} \in \mathrm{T}, \quad \nu = r + jw,$$

then it follows that

$$\begin{aligned} \boldsymbol{x}_k(0, \ldots, (p)_k, \ldots, 0) &= \alpha a_k^{-p} \nu + \bar{\alpha} \bar{a}_k^{-p} \bar{\nu} \\ &= 2r\cos(p\phi + \psi) - 2w\sin(p\phi + \psi). \end{aligned} \tag{7.47}$$

Obviously, it is always possible to choose a phase ψ such that $\boldsymbol{x}_k(0, \ldots, (p)_k, \ldots, 0) \not\to 0$ as $k \to \infty$. For this contradicts the assumption that nD system (7.1) is practically asymptotically stable, we conclude that $\rho(0, \ldots, z_k, \ldots, 0) \neq 0$ for any $z_k \in$T. □

7.3 Practical-Controllability and Practical-Observability

Definitions and necessary and sufficient conditions for practical-controllability and practical-observability will be given in this section.

Because of the linearity of nD system (7.1), we can define the practical-controllability as follows.

Definition 7.2 *nD system (7.1) is said to be practically-controllable if and only if there exist $t_p \geq 0$ for $p = 1, \ldots, n$ such that for any finite $i_k \geq 0$, $k = 1, \ldots, n$, $k \neq p$, any local state $x_p(i_1, \ldots, t_p, \ldots, i_n)$ can be reached from $\mathcal{X}_0 = 0$ by using the input signal sequence $\{\boldsymbol{u}(i_1, \ldots, i_p, \ldots, i_n) \mid 0 \leq i_p < t_p\}$.*

Theorem 7.2 *nD system (7.1) is practically-controllable if and only if the pairs (A_{ii}, B_i) are controllable in the 1D sense for all i $(= 1, \ldots, n)$.*

Proof. When $\boldsymbol{x}(i_1, \ldots, 0_j, \ldots, i_n) = 0$, $j = 1, \ldots, n$, we can get the following relation from the response formula (3.35) and the definition of the state-transition matrix.

$$
\begin{aligned}
&\boldsymbol{x}(i_1, \ldots, t_p, \ldots, i_n) \\
&= \sum_{\substack{(i_1,\ldots,t_p,\ldots,i_n) \\ >(k_1,\ldots,k_p,\ldots,k_n) \\ \geq(0,\ldots,0_p,\ldots,0)}} \Big(\sum_{j=1}^{n} A^{i_1-k_1,\ldots,t_p-k_p,\ldots,i_j-k_j-1,\ldots,i_n-k_n} B^{0,\ldots,1_j,\ldots,0} \Big) \\
&\qquad \cdot \boldsymbol{u}(k_1, \ldots, k_p, \ldots, k_n) \\
&= \sum_{*} \Big(\sum_{j=1}^{n} A^{i_1-k_1,\ldots,t_p-k_p,\ldots,i_j-k_j-1,\ldots,i_n-k_n} B^{0,\ldots,1_j,\ldots,0} \Big) \\
&\qquad \cdot \boldsymbol{u}(k_1, \ldots, k_p, \ldots, k_n) + \\
&\quad + \sum_{k_p=0}^{t_p-1} \Big(\sum_{\substack{j=1 \\ j \neq p}}^{n} A^{0,\ldots,t_p-k_p,\ldots,-1_j,\ldots,0} B^{0,\ldots,1_j,\ldots,0} + \\
&\quad + A^{0,\ldots,t_p-k_p-1,\ldots,0} B^{0,\ldots,1_p,\ldots,0} \Big) \boldsymbol{u}(i_1, \ldots, k_p, \ldots, i_n) \\
&= \sum_{*} \Big(\sum_{j=1}^{n} A^{i_1-k_1,\ldots,t_p-k_p,\ldots,i_j-k_j-1,\ldots,i_n-k_n} B^{0,\ldots,1_j,\ldots,0} \Big) \\
&\qquad \cdot \boldsymbol{u}(k_1, \ldots, k_p, \ldots, k_n) + \\
&\quad + \sum_{k_p=0}^{t_p-1} A^{0,\ldots,t_p-k_p-1,\ldots,0} B^{0,\ldots,1_p,\ldots,0} \boldsymbol{u}(i_1, \ldots, k_p, \ldots, i_n)
\end{aligned} \tag{7.48}
$$

where $*$ denotes the condition

$$
(i_1, \ldots, t_p, \ldots, i_n) > (k_1, \ldots, k_p, \ldots, k_n) \geq (0, \ldots, 0_p, \ldots, 0)
$$

$$
\bigcap (k_1, \ldots, k_p, \ldots, k_n) \neq (i_1, \ldots, \delta_p, \ldots, i_n), \quad \delta_p = 0, 1, \ldots, t_p - 1. \tag{7.49}
$$

Then, in view of Equations (3.32) and (3.36), it is ready to see that

$$
\sum_{k_p=0}^{t_p-1} A^{0,\ldots,t_p-k_p-1,\ldots,0} B^{0,\ldots,1_p,\ldots,0} \boldsymbol{u}(i_1, \ldots, k_p, \ldots, i_n) =
$$

$$
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
A_{pp}^{t_p-1} B_p & A_{pp}^{t_p-2} B_p & \cdots & A_{pp} B_p & B_p \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{u}(i_1, \ldots, 0_p, \ldots, i_n) \\
\boldsymbol{u}(i_1, \ldots, 1_p, \ldots, i_n) \\
\vdots \\
\boldsymbol{u}(i_1, \ldots, t_p-2, \ldots, i_n) \\
\boldsymbol{u}(i_1, \ldots, t_p-1, \ldots, i_n)
\end{bmatrix} \tag{7.50}
$$

The condition (7.49) can be rewritten as follows.

$$
\begin{aligned}
&\bigcup_{\substack{j=1,\ldots,n \\ j \neq p}} \{ (k_1, \ldots, k_p, \ldots, k_n) \mid (0, \ldots, 0_p, \ldots, 0) \leq \\
&\qquad (k_1, \ldots, k_p, \ldots, k_n) \leq (i_1, \ldots, i_j - 1, \ldots, t_p, \ldots, i_n) \}
\end{aligned} \tag{7.51}
$$

It is obvious that on the (time) point of $(k_1, \ldots, k_p, \ldots, k_n) = (i_1, \ldots, t_p, \ldots, i_n)$, the input signal sequence $\{\boldsymbol{u}(k_1, \ldots, k_p, \ldots, k_n)\}$ that satisfies the condition (7.51) has been already known. Without loss of generality, we can assume that these input signals are zero. According to Equation (7.48), therefore, local state $\boldsymbol{x}_p(i_1, \ldots, t_p, \ldots, i_n)$, $p = 1, \ldots, n$ can be reached if and only if there exist appropriate input signal sequence $\boldsymbol{u}(i_1, \ldots, \delta_p, \ldots, i_n)$, $\delta_p = 0, 1, \ldots, t_p - 1$ for some t_p. However, it is easy to see that there exist $\boldsymbol{u}(i_1, \ldots, \delta_p, \ldots, i_n)$, $\delta_p = 0, 1, \ldots, t_p - 1$ for any $\boldsymbol{x}_p(i_1, \ldots, t_p, \ldots, i_n)$ if and only if the matrix

$$
\begin{bmatrix} A_{pp}^{t_p-1} B_p & A_{pp}^{t_p-2} B_p & \cdots & A_{pp} B_p & B_p \end{bmatrix}, \ p = 1, \ldots, n \tag{7.52}
$$

has full row rank. Obviously, this condition is equivalent to controllability of the pair (A_{pp}, B_p) in the 1D sense. □

It is noted that if nD system (7.1) is practically-controllable then $t_p \leq n_p$, $p = 1, \ldots, n$.

Next, the definition and a necessary and sufficient condition for practical-observability are given.

Definition 7.3 *nD system (7.1) is said to be practically-observable if and only if there exist $s_p > 0$ for $p = 1, \ldots, n$ such that whenever $\boldsymbol{u} = 0$ and $\tilde{\mathcal{X}}_0 = 0$ except $\boldsymbol{x}_p(i_1, \ldots, 0_p, \ldots, i_n) \neq 0$ for any finite $i_k \geq 0$, $k = 1, \ldots, n$, $k \neq p$, $\boldsymbol{y}(i_1, \ldots, s_p, \ldots, i_n)$ is not the same as when $\boldsymbol{x}_p(i_1, \ldots, 0_p, \ldots, i_n) = 0$.*

Theorem 7.3 *nD system (7.1) is practically-observable if and only if the pairs (C_i, A_{ii}) is observable in the 1D sense for all i $(= 1, \ldots, n)$.*

Proof. From Equation (3.35) and the assumption of Definition 7.3, we have

$$\begin{aligned} \boldsymbol{y}(i_1, \ldots, s_p, \ldots, i_n) &= C\boldsymbol{x}(i_1, \ldots, s_p, \ldots, i_n) \\ &= C\sum_{j=1}^{n} \sum_{\substack{(i_1,\ldots,s_p\ldots,i_j,\ldots,i_n) \\ \geq(k_1,\ldots,k_p\ldots,k_j,\ldots,k_n) \\ \geq(0,\ldots,0_p,\ldots,0_j,\ldots,0)}} A^{i_1-k_1,\ldots,s_p-k_p,\ldots,i_j,\ldots,i_n-k_n} \\ &\quad \cdot[0 \ \cdots \ \boldsymbol{x}_j^T(k_1, \ldots, k_p, \ldots, 0_j, \ldots, k_n) \cdots 0]^T \\ &= C\sum_{j=1}^{n} A^{0,\ldots,i_j,\ldots,0}[0 \cdots \boldsymbol{x}_j^T(i_1, \ldots, 0_j, \ldots, i_n) \cdots 0]^T \qquad (\text{where } i_p = s_p) \\ &= CA^{0,\ldots,s_p,\ldots,0}[0 \cdots \boldsymbol{x}_p^T(i_1, \ldots, 0_p, \ldots, i_n) \cdots 0]^T \\ &= C_p A_{pp}^{s_p} \boldsymbol{x}_p(i_1, \ldots, 0_p, \ldots, i_n) \end{aligned} \tag{7.53}$$

Let $s_p = n_p - 1$. Then the following relation holds.

$$\begin{bmatrix} \boldsymbol{y}(i_1, \ldots, 0_p, \ldots, i_n) \\ \boldsymbol{y}(i_1, \ldots, 1_p, \ldots, i_n) \\ \vdots \\ \boldsymbol{y}(i_1, \ldots, n_p - 1, \ldots, i_n) \end{bmatrix} = \begin{bmatrix} C_p \\ C_p A_{pp} \\ \vdots \\ C_p A_{pp}^{n_p-1} \end{bmatrix} \boldsymbol{x}_p(i_1, \ldots, 0_p, \ldots, i_n) \tag{7.54}$$

It is now evident that there exists nontrivial solution $\boldsymbol{x}_p(i_1, \ldots, 0_p, \ldots, i_n)$ to Equation (7.54) if and only if the matrix

$$\left[C_p^T \ \ (C_p A_{pp})^T \ \cdots \ (C_p A_{pp}^{n_p-1})^T\right]^T \tag{7.55}$$

has full column rank, which is equivalent to the observability of the pairs of (C_p, A_{pp}) in the 1D sense. □

From 1D system theory and the results of Theorems 7.2 and 7.3, it is trivial to see that a *duality* can be built up between practical-controllability and practical-observability.

7.4 Practical-Stabilizability and Practical-Detectability

7.4.1 Definitions of Practical-Stabilizability and Practical-Detectability

nD system (7.1) will be said to be *practically asymptotically stabilizable*, or simply *practically-stabilizable*, if there exist a local state feedback such that the feedback system (see Figure 7.2) is practically asymptotically stable, and to be *practically detectable* if there exists an asymptotic observer for the local state $\boldsymbol{x}(i_1, \ldots, i_n)$ whose estimate error vanishes as $i_1 + \cdots + i_n \to \infty$ but $(i_1, \ldots, i_n) \in Z_+^{-n}$.

To give the definitions formally, we note that the variables (or coordinates) of nD system (7.1) can be changed by a suitable transformation

$$\boldsymbol{v}(i_1, \ldots, i_n) = T\boldsymbol{x}(i_1, \ldots, i_n) \tag{7.56}$$

where $T \in \mathbf{R}^{\bar{n}\times\bar{n}}$ is a nonsingular matrix (see, e.g., [65]). However, the transformed system

$$\begin{cases} \boldsymbol{v}'(i_1, \ldots, i_n) = \bar{A}\boldsymbol{v}(i_1, \ldots, i_n) + \bar{B}\boldsymbol{u}(i_1, \ldots, i_n) \\ \boldsymbol{y}(i_1, \ldots, i_n) = \bar{C}\boldsymbol{v}(i_1, \ldots, i_n) + \bar{D}\boldsymbol{u}(i_1, \ldots, i_n) \end{cases} \tag{7.57}$$

where

$$\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}, \quad \bar{D} = D$$

has to hold the same nature as system (7.1). For this reason, the transformation matrix T is restricted in the following form [65]:

$$T = T_0 = \text{block diag}(T_1, \ldots, T_n) \tag{7.58}$$

where $T_i \in \mathbf{R}^{n_i \times n_i}$, $i = 1, \ldots, n$, are nonsingular matrices. In this case, we have $Z = T_0 Z T_0^{-1}$. Hence, it is not difficult to verify that all properties for system (7.1), for instance the transfer function matrix, practical-controllability and practical-observability, are consistent with those for the system (7.57).

For an nD system described by the model (7.1) (which may not be practically-controllable and/or practically-observable), we can obtain the Kalman decompositions of (A_{ii}, B_i, C_i), $i = 1, \ldots, n$, if we decide every block element T_i, $i = 1, \ldots, n$, of T_0 by using the Kalman decomposition theorem, respectively (see, e.g., [55]). In other words, applying the Kalman decomposition method, every 1D system corresponding to (A_{ii}, B_i, C_i) can be transformed to another realization where the controllable and observable part, controllable but not observable part, observable but not controllable part and neither controllable nor observable part of state variables can be clearly identified.

As a natural result of the application of Kalman decomposition to nD system (7.1) in the above sense, we can define practical-stabilizability and practical-detectability as follows:

Definition 7.4 *nD system (7.1), or simply the pair (A, B) is said to be practically-stabilizable if and only if the non-controllable part (sub-system) of every (A_{ii}, B_i), $i = 1, \ldots, n$, is stable in the 1D sense.*

Definition 7.5 *nD system (7.1), or simply the pair (A, C) is said to be practically-detectable if and only if the non-observable part (sub-system) of every (A_{ii}, C_i), $i = 1, \ldots, n$, is stable in the 1D sense.*

The rationality of these definitions and associated conditions will become clear in the following sub-sections.

7.4.2 Practical-Stabilization by State Feedback

The following result can be established according to Lemma 7.1:

Theorem 7.4 *nD system (7.1) is practically asymptotically stable if and only if A_{kk}, $k = 1, \ldots, n$, are stable in the 1D sense, i.e., each A_{kk} is devoid of eigenvalues in $\bar{U}$.*

Proof. The proof is obvious from Lemma 7.1 and Theorem 7.1. □

Now consider the practical-stabilizability of nD system (7.1) by the state feedback law

$$\boldsymbol{u}(i_1, \ldots, i_n) = \boldsymbol{v}(i_1, \ldots, i_n) - K\boldsymbol{x}(i_1, \ldots, i_n) \tag{7.59}$$

where $\boldsymbol{v} \in \mathbf{R}^m$ is a new input vector and $K = [K_1\ K_2 \cdots K_n]$, $K_i, i = 1, \ldots, n$, are real feedback gain matrices of appropriate dimensions. Substituting (7.59) into (7.1) yields the following state equation of the resultant closed-loop system

$$\begin{cases} \boldsymbol{x}'(i_1, \cdots, i_n) = A_c\boldsymbol{x}(i_1, \cdots, i_n) + B\boldsymbol{v}(i_1, \ldots, i_n) \\ \boldsymbol{y}(i_1, \ldots, i_n) = (C - DK)\boldsymbol{x}(i_1, \ldots, i_n) + D\boldsymbol{v}(i_1, \ldots, i_n) \end{cases} \tag{7.60}$$

where

$$A_c = A - BK$$

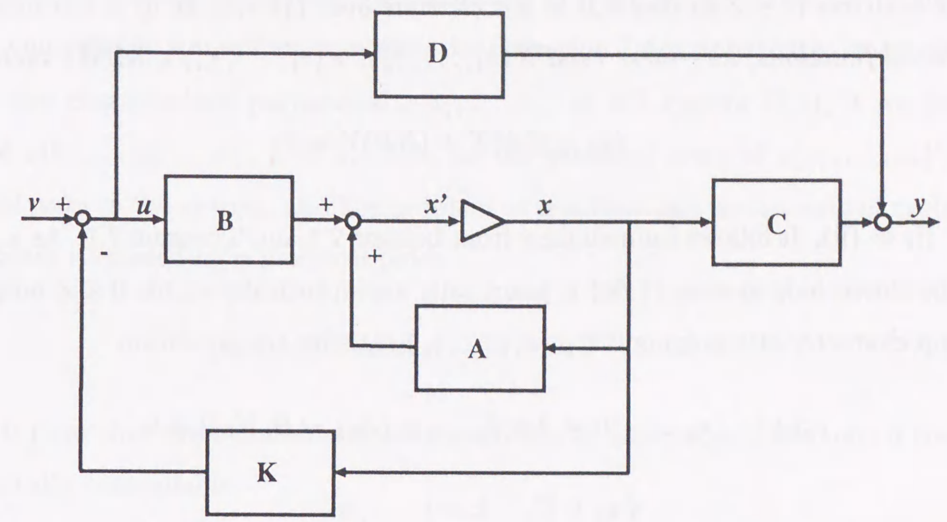


Figure 7.2: Block diagram of the closed-loop system (7.60)

$$= \begin{bmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 & \cdots & A_{1n} - B_1K_n \\ A_{21} - B_2K_1 & A_{22} - B_2K_2 & \cdots & A_{2n} - B_2K_n \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} - B_nK_1 & A_{n2} - B_nK_2 & \cdots & A_{nn} - B_nK_n \end{bmatrix}. \tag{7.61}$$

The block diagram of the feedback system (7.60) is shown in Figure 7.2, and its closed-loop characteristic polynomial reads as

$$\rho_c(z_1, \ldots, z_n) = \det(I - ZA_c). \tag{7.62}$$

The following theorem gives a necessary and sufficient condition for the closed-loop nD system (7.60) to be practically-stable, or nD system (7.1) is practically-stabilizable by the local state feedback (7.59).

Theorem 7.5 *The following conditions are equivalent:*

(i) *nD system (7.1) is practically-stabilizable by the local state feedback (7.59);*

(ii) *All the pairs (A_{kk}, B_k), $k = 1, \ldots, n$, are stabilizable in the (1D) sense, i.e.,*

$$\operatorname{rank}[I_{n_k} - z_kA_{kk}\ \ z_kB_k] = n_k \quad \forall z_k \in \bar{U},\ k = 1, \ldots, n \tag{7.63}$$

(iii) *The matrices* $(I - ZA)$ *and* ZB *is left coprime over the ring* $\tilde{\mathbf{H}}$ *of practically-stable rational functions, i.e., there exist* $X(z_1, \dots, z_n)$, $Y(z_1, \dots, z_n) \in \mathbf{M}(\tilde{\mathbf{H}})$ *such that*

$$(I_{\tilde{n}} - ZA)X + (ZB)Y = I. \tag{7.64}$$

Proof: (i) $\Leftrightarrow$ (ii). It follows immediately from Lemma 7.1 and Theorem 7.1. As a matter of fact, the closed-loop system (7.60) is practically asymptotically stable if and only if the closed-loop characteristic polynomial $\rho_c(z_1, \dots, z_n)$ satisfies the condition

$$\rho_c(0, \dots, z_k, \dots, 0) = \det(I_{m_k} - z_k(A_{kk} - B_k K_k)) \neq 0$$

$$\forall\, z_k \in \bar{U}, \quad k = 1, \dots, n. \tag{7.65}$$

According to the 1D system theory, this is equivalent to the condition given by Equation (7.63).

(ii) $\Leftrightarrow$ (iii). From Theorem 6.3 (or [116]), (iii) is equivalent to the condition

$$\operatorname{rank}[I_{\tilde{n}} - ZA \ \ ZB](0, \dots, z_k, \dots, 0) = \tilde{n} \quad \forall z_k \in \bar{U} \tag{7.66}$$

Since

$$\begin{aligned}
&\operatorname{rank}[I_{\tilde{n}} - ZA \ \ ZB](0, \dots, z_k, \dots, 0) \\
&= \operatorname{rank}\begin{bmatrix} I_{n_1} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_{n_2} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ -z_k A_{k1} & -z_k A_{k2} & \cdots & I_{n_k} - z_k A_{kk} & \cdots & -z_k A_{kn} & z_k B_k \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_{n_n} & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} I_{(\tilde{n}-n_k)} & 0 & 0 \\ 0 & I_{n_k} - z_k A_{kk} & z_k B_k \end{bmatrix},
\end{aligned} \tag{7.67}$$

it is easy to see the condition (7.66) is equivalent to (ii). □

The results of Theorem 7.5 reveals that the practical-stabilization problem of nD system (7.1) by state feedback (7.59) is in fact equivalent to the stabilization problems of n 1D systems described by $(A_{ii},\ B_i)$, $i = 1, \dots, n$. Therefore, the feedback gain matrices K_i, $i = 1, \dots, n$, can be determined entirely by 1D methods, and this coincides with the consequence obtained by algebra approach in Chapter 6 (see also [116]). Moreover, since

it is well-known that a 1D system corresponding to $(A_{ii},\ B_i)$ is stabilizable if and only if its non-controllable sub-system is stable, the definition 7.4 is apparently reasonable.

For the characteristic polynomial $\rho(z_1, \dots, z_n)$ of nD system (7.1), if we define the zeros of $\rho(0, \dots, z_k, \dots, 0)$, $k = 1, \dots, n$, as the *practical zeros* of $\rho(z_1, \dots, z_n)$ and the *practical poles* of the system, and the problem of *practical pole assignment* of system (7.1) as to locate its closed-loop practical poles

$$\{z_k \mid \rho_c(0, \dots, z_k, \dots, 0) = 0, k = 1, \dots, n\}, \tag{7.68}$$

then it is clear that these practical poles are arbitrarily assignable if and only if the system is practically-controllable.

7.4.3 Asymptotic State Observer in the Practical Sense

We first give necessary and sufficient conditions for practical-detectability as follows.

Theorem 7.6 *The following conditions are equivalent:*

(i) nD *system* (7.1) *is practically-detectable;*

(ii) *All the pairs* $(A_{kk},\ C_k)$, $k = 1, \dots, n$, *is detectable in the* (1D) *sense, i.e.,*

$$\operatorname{rank}\begin{bmatrix} C_k \\ I_{n_k} - z_k A_{kk} \end{bmatrix} = n_k \quad \forall z_k \in \bar{U},\ k = 1, \dots, n \tag{7.69}$$

(iii) *The matrices* $(I - ZA)$ *and* C *is right coprime over the ring* $\tilde{\mathbf{H}}$ *of practically-stable rational functions (see Chapter 6 or* [116]*), i.e., there exist* $U(z_1, \dots, z_n)$, $V(z_1, \dots, z_n) \in \mathbf{M}(\mathbf{H})$ *such that*

$$U(I_{\tilde{n}} - ZA) + VC = I \tag{7.70}$$

Proof: It is easy to see that a duality can be built up between practical-stabilizability and practical-detectability, and the proof can be shown in the same way as for Theorem 7.5. □

The results of Theorem 7.6 implies that observer for nD system (7.1) in the practical sense can be constructed by using 1D approaches.

Consider an nD observer described by

$$\begin{cases} \tilde{x}'(i_1,\ldots,i_n) = A\tilde{x}(i_1,\ldots,i_n) + Bu(i_1,\ldots,i_n) \\ \qquad\qquad\qquad + F\{y(i_1,\ldots,i_n) - \tilde{y}(i_1,\ldots,i_n)\} \\ \tilde{y}(i_1,\ldots,i_n) = C\tilde{x}(i_1,\ldots,i_n) \end{cases} \tag{7.71}$$

where $\tilde{x}(i_1,\ldots,i_n) \in \mathbf{R}^{\bar{n}}$ is the state vector of the observer, $\tilde{y}(i_1,\ldots,i_n) \in \mathbf{R}^l$ is the output vector, $F = [F_1^T \cdots F_n^T]^T$ is a real constant matrix with suitable dimensions. To estimate the local state $x(i_1,\ldots,i_n)$, we should choose F such that the error

$$e(i_1,\ldots,i_n) = x(i_1,\ldots,i_n) - \tilde{x}(i_1,\ldots,i_n) \tag{7.72}$$

can be properly controlled. From Equations (7.1) and (7.71), the error $\tilde{x}(i_1,\ldots,i_n)$ obeys the equation

$$e'(i_1,\ldots,i_n) = (A - FC)e(i_1,\ldots,i_n). \tag{7.73}$$

According to the results of Theorem 7.6, there exists some F such that $I - Z(A - FC)$ is practically asymptotically stable if and only if (A, C) is practically-detectable, which means that the error $e(i_1,\ldots,i_n) \to 0$ as $i_1 + \cdots + i_n \to \infty$ but $(i_1,\ldots,i_n) \in Z_+^{-n}$. Moreover, it is also obvious that the practical zeros of $\det(I - Z(A - FC))$ can be arbitrarily assigned if and only if (A, C) is practically-observable.

7.5 Relationship Between Practical-BIBO Stability and Practical Internal Stability

Based on the results obtained above, the relationship between the practical-BIBO stability and the practical internal stability can be clarified.

Theorem 7.7 *For nD system (7.1), the following conditions are equivalent:*

(i) *nD system (7.1) is practically asymptotically stable;*

(ii) *nD system (7.1) is practically-stabilizable, practically-detectable and its transfer function (7.7) is practically-BIBO stable.*

Proof. (i) $\Rightarrow$ (ii). Suppose that nD system (7.1) is practically asymptotically stable. From Theorem 7.1, $\rho(z_1,\ldots,z_n) = \det(I - ZA)$ satisfies the condition

$$\rho(0,\ldots,z_k,\ldots,0) \neq 0, \quad \forall z_k \in \bar{U}, \quad k = 1,\ldots,n. \tag{7.74}$$

Further, Equations (7.64) and (7.70) are satisfied by setting

$$Y(z_1,\ldots,z_n) = V(z_1,\ldots,z_n) = 0, \tag{7.75}$$

$$X(z_1,\ldots,z_n) = U(z_1,\ldots,z_n) = (I - ZA)^{-1}. \tag{7.76}$$

By Theorems 7.5 and 7.6, therefore, nD system (7.1) is practically-stabilizable and practically-detectable. Noting that the denominator of the transfer function (7.7), i.e., $\det(I - ZA)$ is devoid of zeros in $\bar{U}$, the proof is completed.

(ii) $\Rightarrow$ (i). Suppose that nD system (7.1) is practically-BIBO stable, practically-stabilizable and practically-detectable. According to Theorems 7.5 and 7.6, then, there exist matrices X, Y, U, $V \in \mathbf{M}(\tilde{\mathbf{H}})$ satisfying Equations (7.64) and (7.70). Then, from Equations (7.64) and (7.70) we get

$$X + (I - ZA)^{-1}ZBY = (I - ZA)^{-1} \tag{7.77}$$

$$U + VC(I - ZA)^{-1} = (I - ZA)^{-1} \tag{7.78}$$

Substituting Equation (7.78) into Equation (7.77) and using the result of Equation (7.7) yield

$$(I - ZA)^{-1} = X + UZBY - VDY + VGY \tag{7.79}$$

By the assumption that nD system (7.1) is practically-BIBO stable, i.e., $G \in \mathbf{M}(\tilde{\mathbf{H}})$, the right-hand side of Equation (7.79) belongs to $\mathbf{M}(\tilde{\mathbf{H}})$.

Consider now the free state evolution of nD system (7.1) for any initial global state $\tilde{\mathcal{X}}_0$ associated with a bounded sequence of local states $\{x(i_1,\ldots,i_n) \mid \sum_{j=1}^{n} i_j = 0\}$. By Equation (7.3), the state evolution obeys

$$x(z_1,\ldots,z_n) = (I - ZA)^{-1}\tilde{\mathcal{X}}_0. \tag{7.80}$$

In view of that $(I - ZA)^{-1} \in \mathbf{M}(\tilde{\mathbf{H}})$ and the proof for the sufficiency of Theorem 7.1, we conclude that nD system (7.1) is practically asymptotically stable. □

7.6 A Connection Between State-Space and Doubly Coprime MFD on $\tilde{\mathbf{H}}$

In this section, a matrix $A \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ is said to be practically-stable if $\det(I - ZA) \in \tilde{\mathbf{J}}$. By Theorem 7.5, then, there exists $K \in \mathbf{R}^{m\times\tilde{n}}$ such that $A - BK$ is practically-stable whenever $(A, B) \in \mathbf{R}^{\tilde{n}\times\tilde{n}} \times \mathbf{R}^{\tilde{n}\times m}$ is practically-stabilizable. Moreover, by Theorem 7.6, there is $F \in \mathbf{R}^{\tilde{n}\times l}$ such that $A - FC$ is practically-stable if (C, A) is practically-detectable.

A connection between state-space and doubly coprime MFD on $\tilde{\mathbf{H}}$ is now shown by the following theorem.

Theorem 7.8 *Suppose that the transfer matrix of nD system (7.1) is*

$$G(z_1, \ldots, z_n) = C(I - ZA)^{-1}ZB \in \mathbf{M}(\mathbf{G})$$

where (A, B) is practically-stabilizable and (A, C) is practically-detectable. Choose K and F such that $A - BK$ and $A - FC$ are practically-stable. Define

$$\tilde{N}(z_1, \ldots, z_n) = C[I - Z(A - BK)]^{-1}ZB \tag{7.81}$$

$$\tilde{D}(z_1, \ldots, z_n) = I - K[I - Z(A - BK)]^{-1}ZB \tag{7.82}$$

$$\tilde{U}(z_1, \ldots, z_n) = K[I - Z(A - FC)]^{-1}ZF \tag{7.83}$$

$$\tilde{V}(z_1, \ldots, z_n) = I + K[I - Z(A - FC)]^{-1}ZB \tag{7.84}$$

$$\bar{D}(z_1, \ldots, z_n) = I - C[I - Z(A - FC)]^{-1}ZF \tag{7.85}$$

$$\bar{N}(z_1, \ldots, z_n) = C[I - Z(A - FC)]^{-1}ZB \tag{7.86}$$

$$\bar{V}(z_1, \ldots, z_n) = I + C[I - Z(A - BK)]^{-1}ZF \tag{7.87}$$

$$\bar{U}(z_1, \ldots, z_n) = K[I - Z(A - BK)]^{-1}ZF \tag{7.88}$$

then

1) *all matrices defined in (7.81)–(7.88) are practically-stable.*

2) *$\tilde{D}(z_1, \ldots, z_n)$ and $\bar{D}(z_1, \ldots, z_n)$ are nonsingular.*

3)

$$\begin{aligned} G(z_1, \ldots, z_n) &= \tilde{N}(z_1, \ldots, z_n)\tilde{D}^{-1}(z_1, \ldots, z_n) \\ &= \bar{D}^{-1}(z_1, \ldots, z_n)\bar{N}(z_1, \ldots, z_n) \end{aligned} \tag{7.89}$$

4)

$$\begin{bmatrix} \tilde{V} & \tilde{U} \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} \tilde{D} & -\bar{U} \\ \tilde{N} & \bar{V} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{7.90}$$

Proof.

1). It is obvious from the definition of practically-stable matrices (i.e., matrices having entries in $\tilde{\mathbf{H}}$).

2). Noting the relation $\det(I - XY) = \det(I - YX)$, the regularity of $\tilde{D}(z_1, \ldots, z_n)$ can be shown as follows:

$$\begin{aligned} &\det \tilde{D}(z_1, \ldots, z_n) \\ &= \det\{I - K[I - Z(A - BK)]^{-1}ZB\} \\ &= \det\{I - [I - Z(A - BK)]^{-1}ZBK\} \\ &= \det\{[I - Z(A - BK)]^{-1}(I - ZA)\} \\ &= \frac{\det(I - ZA)}{\det[I - Z(A - BK)]} \not\equiv 0 \end{aligned} \tag{7.91}$$

Similarly,

$$\det \bar{D}(z_1, \ldots, z_n) = \frac{\det(I - ZA)}{\det[I - Z(A - FC)]} \not\equiv 0 \tag{7.92}$$

can be shown.

3) and 4) can be proved in the same way as in [77]. For brevity, we just show $\tilde{U}\tilde{N} + \tilde{V}\tilde{D} = I$ in the following.

$$\begin{aligned} &\tilde{U}(z_1, \ldots, z_n)\tilde{N}(z_1, \ldots, z_n) + \tilde{V}(z_1, \ldots, z_n)\tilde{D}(z_1, \ldots, z_n) \\ &= K[I - Z(A - FC)]^{-1}ZFC[I - Z(A - BK)]^{-1}ZB \\ &\quad + \{I + K[I - Z(A - FC)]^{-1}ZB\}\{I - K[I - Z(A - BK)]^{-1}ZB\} \\ &= K[I - Z(A - FC)]^{-1}ZFC[I - Z(A - BK)]^{-1}ZB + \\ &\quad + I + K[I - Z(A - FC)]^{-1}ZB - K[I - Z(A - BK)]^{-1}ZB \end{aligned}$$

$$-K[I-Z(A-FC)]^{-1}ZBK[I-Z(A-BK)]^{-1}ZB$$

$$= I + K[I - Z(A - FC)]^{-1}\{ZFC - ZBK$$

$$+[I - Z(A - BK)] - [I - Z(A - FC)]\}[I-Z(A-BK)]^{-1}ZB$$

$$= I \tag{7.93}$$

□

Similarly as in [77], the results of Theorem 7.8 can be easily generalized to the case of $G(z_1, \ldots, z_n) = C(I - ZA)^{-1}ZB + D \in \mathbf{M}(\mathbf{G})$.

Moreover, it should be clear that by the results of Chapter 6 (see also [116]), the class of all practically-stabilizing compensators for nD system (7.1) can be explicitly given by utilizing the double coprime relation (7.90).

7.7 Summary

The concept of practical internal stability, or practical asymptotic stability, has been introduced based on nD Roesser state-space model, and a necessary and sufficient condition for the stability has been shown. This condition reveals that practical internal stability is equivalent to the stabilities of n 1D systems. Namely, if $n(z_1, \ldots, z_n)/d(z_1, \ldots, z_n)$ is the transfer function of an (SISO) nD system described by nD Roesser state-space model, then this system is practically asymptotically stable if and only if $d(0, \ldots, z_k, \ldots, 0) \neq 0$, $\forall |z_k| \leq 1$, $k = 1, \ldots, n$. In contrast with this, when $n = 2$, the system $n(z_1, z_2)/d(z_1, z_2)$ is asymptotically stable in the sense of conventional internal stability if and only if $d(z_1, z_2) \neq 0$, $\forall (z_1, z_2) \in \bar{U}^2$ [2, 40, 41]. By these results, it can be seen that the conventional definition of internal stability is also unnecessarily restrictive for many practical applications, just as that shown by Agathoklis and Bruton [1] for practical-BIBO stability and conventional-BIBO stability.

Then, the notions of practical-controllability and practical-observability have been established and associated necessary and sufficient conditions have been derived. By comparing these results with those in [24, 66], we note that practical-controllability (respectively practical-observability) is equivalent to r-reachability for any n and to model controllability (model observability) for $n = 2$.

We have also shown the solvability of the problems of practical-stabilization by local state feedback and construction of asymptotic state observer in the practical sense. Based on these results, it has been clarified that a practical-BIBO stable nD system is also practically internally stable if its transfer function has a practically-detectable and practically-stabilizable state-space realization. Similarly to the 1D case [77], a connection between state-space and doubly coprime MFD on $\tilde{\mathbf{H}}$ has been given, and by this result and those obtained in the previous chapter the class of practically-stabilizing compensators of nD system (7.1) can be explicitly parameterized.

It should also be noted, just as the consequence obtained by algebra approach in Chapter 6, all the problems considered in this chapter can be formulated to the corresponding problems of n 1D systems, therefore can be resolved by using 1D methods.

As very interesting applications of nD system theory, it is now well-known that iterative learning control systems and linear multipass processes can be described by 2D system model [16, 44, 72, 83]. A common feature of these systems is obviously that while the iterations are not subjected to any boundary condition, the iterated passes are usually bounded on a finite discrete-time interval. It is, therefore, expected that the results obtained here and in the previous chapter might supply a basis for establishing a unified design method for these systems.

Chapter 8

Concluding Remarks

According to the two-fold objective of the research stated in Chapter 1, this thesis systematically treated

- the output feedback stabilization and servo problems for 2D systems by means of MFD algebraic and Gröbner basis approaches (Chapters 4 and 5); and
- some fundamental properties and feedback control problems for nD systems in terms of the practical stability theory [1, 119, 122] by MFD algebraic approach as well as state-space approach (Chapter 6 and Chapter 7).

The main contributions can be briefly summarized in three aspects as follows:

1. Output feedback stabilizability and stabilization algorithms for 2D systems (Chapter 4)

Alternative tests for 2D output feedback stabilizability and a new construction procedure for 2D stable closed-loop polynomial $s(v, w)$ have been proposed. By the proposed methods, the problems can be easily reduced to the 1D case and solved by using well-developed 1D algorithms.

The so-called "Rabinowitsch trick", a technique ever used in the well-known Hilbert's Nullstellensatz (see e.g. [124]), has been generalized in some senses to the case of modules over polynomial ring (Lemmas 4.1, 4.2). These results eventually leaded to two new constructive solution algorithms, Algorithms 4.1 and 4.2, for the 2D unilateral polynomial matrix equation

$$D(v, w)X(v, w) + N(v, w)Y(v, w) = C(v, w) \tag{8.1}$$

where $D(v,w)$, $N(v,w)$ are given, $X(v,w)$, $Y(v,w)$ and $C(v,w)$ are to be found and $C(v,w)$ has to be stable. In particular, Algorithm 4.2 shows that the main task we should do in order to solve the equation is just finding the Gröbner bases of certain modules characterized in terms of the columns of the matrix $[D(v,w)\ N(v,w)]$ and the stable polynomial $s(v,w)$. This algorithm is obviously superior to the existing methods at some points, as it requires neither computation of any minors or zeros, nor estimation of any degree, but yet has a general form for $C(v,w)$.

By these results, then, the construction and parametrization of stabilizing compensators can be carried out in the standard way [49, 103].

2. Skew Ω-primeness and servo problems for 2D systems

(Chapter 5)

We first introduced the concept of zero skew Ω-primeness for two Ω-stable rational matrices, where Ω is a subdomain of $\mathbf{C}^2$ containing the origin, and Ω-stable rational matrix are those whose entries belong to the ring of 2D rational functions with no poles in Ω. Necessary and sufficient conditions for skew Ω-primeness have been derived.

Then, we examined the construction of solutions $U(v,w)$, $V(v,w)$ to the equation

$$D(v,w)U(v,w) + V(v,w)N(v,w) = I \tag{8.2}$$

when $D(v,w)$ and $N(v,w)$ are two zero skew Ω-prime polynomial matrices, where $U(v,w)$ and $V(v,w)$ are Ω-stable rational matrices. Solvability conditions and solution procedure have been proposed.

Necessary and sufficient conditions for the solvability of the following bilateral polynomial matrix equation have also been given.

$$D(v,w)X(v,w) + Y(v,w)N(v,w) = C(v,w) \tag{8.3}$$

where $D(v,w)$, $N(v,w)$ and $C(v,w)$ are given 2D polynomial matrices, and $X(v,w)$, $Y(v,w)$ are 2D polynomial matrix solution to be found.

Based on the above results, the asymptotic and deadbeat tracking and regulation problems for 2D systems have been solved in a unified way.

3. Design of practically-stable nD feedback systems

(Chapter 6 and Chapter 7)

In this part, starting from the concept of practical-BIBO stability introduced by Agathoklis and Bruton [1], several fundamental control problems have been investigated for nD systems whose input and output signals are unbounded in, at most, one dimension.

First of all, by using algebraic approach, we solved the feedback practical-stabilization problem of nD systems. A constructive algorithm has been given for solving Bezout equation over the ring of practically-stable nD rational functions. A necessary and sufficient condition for an nD system to be practically-stabilizable has been derived and the class of all nD practically-stabilizing compensators have been parametrized.

Then, the concept of practical internal stability, or practical asymptotic stability, has been introduced based on nD Roesser state-space model, and a necessary and sufficient condition for the stability has been shown. This condition reveals that practical internal stability is equivalent to the stabilities of n 1D systems, which implies that the conventional definition of internal stability for 2D system [2, 40, 41] is also unnecessarily restrictive for many practical applications, just as that shown by Agathoklis and Bruton [1] for practical-BIBO stability and conventional-BIBO stability.

Next, the notions of practical-controllability and practical-observability have been established and associated necessary and sufficient conditions have been derived. We have also shown the solvability conditions for practical-stabilization by local state feedback and construction of asymptotic state observer in the practical sense. In terms of these results, the relationship between practical-BIBO stability and internal practical stability has been clarified under the introduced concepts of practical-detectability and practical-stabilizability.

Similarly to the 1D case [77], a connection between state-space and doubly coprime MFD on the ring of practically-stable rational functions has be given.

The obtained results make it clear that all the above-mentioned nD control problems in the practical sense of [1] can be formulated to the corresponding 1D problems. This property is of special significance for it implies that some control problems, which may be insolvable under the concept of conventional stability, can be solved under the (less restrictive) practical stability and meanwhile by just using 1D methods.

On the other hand, we would like to remark, as future topics, some further possibilities and problems.

- **Extension of the results of Chapter 4 to the $n > 2$ case**

In Chapter 4, we considered the test of output feedback stabilizability and the construction of a stable closed-loop polynomial for 2D systems, where we also commented that these results can be extended to the nD ($n > 2$) case under certain conditions. Now, we explain this possibility and related problems a little more concretely.

In the standard way as for 2D systems, it is ready to see that stabilization problem of an nD system given by $D^{-1}(z_1,\ldots,z_n)N(z_1,\ldots,z_n)$ with $D, N \in \mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$ is equivalent to the solvability of the unilateral matrix equation

$$D(z_1,\ldots,z_n)X(z_1,\ldots,z_n)+N(z_1,\ldots,z_n)Y(z_1,\ldots,z_n)=C(z_1,\ldots,z_n) \quad (8.4)$$

where $X,\ Y, C \in \mathbf{M}(\mathbf{R}[z_1,\ldots,z_n])$ and C is stable, i.e, devoid of zeros in $\bar{U}^n$.

Let $a_i \in \mathbf{R}[z_1,\ldots,z_n]$, $i = 1,\ldots,\beta$ correspond to all the maximal order minors of $[D(z_1,\ldots,z_n)\ N(z_1,\ldots,z_n)]$, and $\mathcal{V}(\mathcal{I})$ the variety of the ideal $\mathcal{I}$ generated by these minors.

If $\mathcal{I}$ is zero-dimensional, i.e., $\mathcal{V}(\mathcal{I})$ is a finite set (or the minors a_i, $i = 1,\ldots,\beta$, have finitely many common zeros), then it is easy to show that, in the same way for the 2D case in [10, 49], we can construct a stable polynomial $s(z_1,\ldots,z_n)$ vanishing on $\mathcal{V}(\mathcal{I})$ (by explicitly computing all the common zeros), and further have some positive integer r such that

$$D(z_1,\ldots,z_n)X(z_1,\ldots,z_n)+N(z_1,\ldots,z_n)Y(z_1,\ldots,z_n)=s^r(z_1,\ldots,z_n)I \quad (8.5)$$

if and only if

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^n = \emptyset. \quad (8.6)$$

Then, the main problems we are interested in are:

(a) how to construct such a closed-loop polynomial $s(z_1,\ldots,z_n)$ without explicit computation of the common zeros;

(b) how to test the stabilizability condition (8.6);

(c) how to solve the equation (8.5) without estimation of r.

Under the assumption that $\mathcal{I}$ is of zero dimension, it is easy to generalize the methods for the 2D case proposed in Section 4.2 to the $n > 2$ cases.

Consider $\mathcal{V}(\mathcal{I})$ as the solution set of a system of polynomial equations, i.e, $a_i(z_1,\ldots,z_n) = 0$, $i = 1,\ldots,\beta$, which will be denoted in the form

$$\mathcal{V}(\mathcal{I}) = \{a_1(z_1,\ldots,z_n)=0,\cdots,a_\beta(z_1,\ldots,z_n)=0\} \quad (8.7)$$

Using Method 6.10 of [23] to compute the Gröbner bases of $\mathcal{I}$ with respect to the purely lexicographical orderings with $z_n >_T \cdots >_T z_{k+1} >_T z_{k-1} >_T \cdots z_1 >_T z_k$, $k = 1,\ldots,n$, we get

$$\{a_1(z_1,\ldots,z_n)=0,\cdots\cdots,a_r(z_1,\ldots,z_n)=0\}$$
$$\Updownarrow$$
$$\{g_1(z_1)=0, g_{12}(z_1,z_2)=0,\cdots,g_{1n}(z_1,\ldots,z_n)=0\} \qquad z_n >_T \cdots >_T z_2 >_T z_1$$
$$\Updownarrow$$
$$\{g_2(z_2)=0, g_{22}(z_2,z_1)=0,\cdots,g_{2n}(z_2,z_1,\ldots,z_n)=0\} \qquad z_n >_T \cdots >_T z_1 >_T z_2$$
$$\Updownarrow$$
$$\cdots\cdots$$
$$\Updownarrow$$
$$\{g_n(z_n)=0, g_{n2}(z_n,z_1)=0,\cdots,g_{nn}(z_n,z_1,\ldots,z_{n-1})=0\} \quad z_{n-1} >_T \cdots >_T z_1 >_T z_n$$

where $g_i(z_i)$, $i = 1,\ldots,n$, are 1D polynomials, and the others are nD ($n \geq 2$) polynomials. (In fact, Method 6.11 of [23] can be applied to compute only the 1D polynomials $g_j(z_j)$, $j = 1,\ldots,n$, more efficiently.) It is then obvious that if $(z_1^0,\ldots,z_n^0) \in \mathcal{V}(\mathcal{I})$, i.e.,

$$a_i(z_1^0,\ldots,z_n^0)=0, \quad i = 1,\ldots,r, \quad (8.8)$$

we have

$$g_j(z_j^0) = 0, \quad j = 1, \dots, n \tag{8.9}$$

Following the very same idea adopted for the 2D case in Section 4.2, we can carry out, by employing 1D methods [69, 78], the decomposition

$$g_j(z_j) = g_{js}(z_j)g_{ju}(z_j), \quad j = 1, \dots, n \tag{8.10}$$

such that $g_{js}(z_j)$ is stable and $g_{ju}(z_j)$ has no stable zeros, $j = 1, \dots, n$. Then, we can construct the stable polynomial

$$s(z_1, \dots, z_n) = \prod_{j=1}^{n} g_{js}(z_j) \tag{8.11}$$

and show it vanishes on $\mathcal{V}(\mathcal{I})$ if the condition (8.6) is satisfied. Further, it is also easy to see that

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^n \subset \Gamma \stackrel{\triangle}{=} \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid g_{ju}(z_j) = 0, j = 1, \dots, n\} \subset \bar{U}^n \tag{8.12}$$

and that the condition (8.6) can be represented as

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^n = \mathcal{V}(\mathcal{I}) \cap \Gamma = \emptyset. \tag{8.13}$$

which means the zero coprimeness of $a_1(z_1, \dots, z_n), \cdots, a_\beta(z_1, \dots, z_n)$, $g_{1u}(z_1)$, $g_{2u}(z_2)$, $\cdots$, $g_{nu}(z_n)$.

In view of the above arguments, it should be clear that the consequences of Theorem 4.1 (except the statement (iv)) and Theorem 4.2 are also true for the nD ($n > 2$) case if $\mathcal{I}$ is zero-dimensional, which solve the problem (b) and (a), respectively.

Moreover, as we have commented after Algorithm 4.1, under the assumption of minor coprimeness on certain matrices (see Section 4.3 for detail), the results obtained in Section 4.3 hold for the $n > 2$ case as well. In this case, therefore, the problem (c) can be solved and the conclusion of the statement (iv) of Theorem 4.1 can be generalized.

However, for the $n > 2$ case, the variety of $\mathcal{I}$ is generally surface or manifold, so the condition for $\mathcal{I}$ to be zero-dimensional seems restrictive in general for an nD

system. Therefore, finding more general construction algorithm of stable closed-loop polynomial for the $n > 2$ case will be an interesting problem in future research. Further, due to the impossibility of getting the minor coprime factorization of two specified nD ($n > 2$) polynomial matrices, the above-mentioned generalization of the results of Section 4.3 under the assumption of minor coprimeness is trivial. Since this condition is not necessary for the nD stabilization problem, we believe that a generalization without this limitation would not be very difficult and in fact the study on this problem is in progress.

- **Practical applications of nD system control theory**

As very interesting applications of nD system theory, it has been reported that iterative learning control systems and linear multipass processes can be described by 2D system model and analyzed by 2D stability theory [16, 44, 83]. It has been shown in these works that 2D system theory offers a mathematical model to characterize the entire dynamics involved in these systems, and 2D stability theory provides a very useful approach to show the convergences of these systems. In [16, 83], however, since no general design methods are available, the systems are only synthesized by using controllers of very restricted types. In other words, the systems are designed in the manner: first restrict the controller to be used to a simple type so that the involved parameters can be determined easily, and then see if it is possible to make the constructed closed-loop system satisfy the specified requirement by this simple controller. Since there is no such standard that the controller can be reformed if the previously given type failed, this kind of design procedure is not generally applicable except for some simple cases.

In Chapter 5 we generally solved the asymptotic tracking problem of 2D systems, so by these results it is possible to construct a general and unified design procedure, without *a priori* restrictions on the structure of controllers, for iterative learning systems and linear multipass processes. However, as mentioned previously, a common feature of these systems is that while there is no boundary restriction on the number of iterations, the iterated dynamic passes are always restricted on a finite discrete-time interval. This fact implies that we may establish much simpler design methods for these systems based on the results obtained in Chapters 6 and 7 for practical asymptotic stability and design of practically-stable nD feedback systems. Establishing such a design procedure and applying it to practical control systems are main objectives of current investigation [111].

References

[1] P. Agathoklis and L. Bruton. Practical-BIBO stability of n-dimensional discrete systems. *IEE Proc. G. Electron. Circuits & Systems*, 6:236–242, 1983.

[2] A. R. E. Ahmed. On the Stability of Two-Dimensional Discrete Systems. *IEEE Trans. Automat. Control*, 25(3):551–552, 1979.

[3] B. D. O. Anderson and E. I. Jury. Stability of Multidimensional Digital Filters. *IEEE Trans. Automat. Control*, 21(2):300–304, 1974.

[4] S. Attasi. Systèms linearires homogènes à deux indices. *Rapport LABORIA*, 31, 1973.

[5] S. Basu. New Results on Stable Multidimensional Polynomials—Part III : State-Space Interpretations. *IEEE Trans. Circuits and Systems*, 38(7):755–768, 1991.

[6] S. Basu and A. Fettweis. New Results on Stable Multidimensional Polynomials—Part II : Discrete cases. *IEEE Trans. Circuits and Systems*, 34:1258–1270, 1987.

[7] C. A. Berenstein and D. C. Struppa. On explicit solutions to the Bezout equation. *Syst. & Control Letters*, 4:33–39, 1984.

[8] C. A. Berenstein and D. C. Struppa. Small degree solutions for the polynomial Bezout equation. *Linear Algebra Appl.*, 98:41–55, 1988.

[9] M. Bisiacco. State and output feedback stabilizability of 2-D systems. *IEEE Trans. Circuits and Systems*, 32:1246–1254, 1985.

[10] M. Bisiacco, E. Fornasini, and G. Marchesini. On some connections between BIBO and internal stability of two-dimensional filters. *IEEE Trans. Circuits and Systems*, 32:948–953, 1985.

[11] M. Bisiacco, E. Fornasini, and G. Marchesini. Causal 2D compensators: Stabilization Algorithms for Multivariable 2D Systems. In *Proc. 25th Conf. on Decision and Control*, pages 2171–2174, 1986.

[12] M. Bisiacco, E. Fornasini, and G. Marchesini. Controller Design for 2D Systems. In C. I. Byrnes and A. Lindquist, editors, *Frequency Domain and State Space Methods for Linear Systems*, pages 99–113, North Holland: Elservier, 1986.

[13] M. Bisiacco, E. Fornasini, and G. Marchesini. Linear Algorithms For Computing Closed Loop Polynomials of 2D Systems. In *Proc. of IEEE Symp. on Circ. and Syst.*, pages 345–348, Helsinki, Finland, 1988.

[14] M. Bisiacco, E. Fornasini, and G. Marchesini. 2D Systems Feedback Compensation: An Approach Based on Commutative Linear Transformations. *Linear Algebra Appl.*, 121:135–150, 1989.

[15] M. Bisiacco, E. Fornasini, and G. Marchesini. Dynamic Regulation of 2D Systems: A State-Space Approach. *Linear Algebra Appl.*, 122/123/124:195–218, 1989.

[16] F. M. Boland and D. H. Owens. Linear multipasses: a two-dimensional interpretation. *IEE Proc.*, 127, Pt. D-5:189–193, 1980.

[17] N. K. Bose. A criterion to determine if two multivariable polynomials are relatively prime. *Proc. IEEE*, 60:134–135, 1972.

[18] N. K. Bose. An algorithm for g.c.f. extraction from two multivariable polynomials. *Proc. IEEE*, 64:185–186, 1976.

[19] N. K. Bose. Problems and Progress in Multidimensional Systems Theory. *Proc. IEEE*, 65(6):824–840, 1977.

[20] N. K. Bose. *Applied Multidimensional Systems Theory.* Van Nostrand Reinhold, N.Y., 1982.

[21] N. K. Bose, editor. *Multidimensional Systems Theory.* Dordrecht: Reidel, N.Y., 1985.

[22] W. D. Brownawell. Bounds for the degrees in the Nullstellensatz. *Annals of Mathematics*, 126:577–591, 1987.

[23] B. Buchberger. Gröbner bases: An Algorithmic Method in Polynomial Ideal Theory. In N. K. Bose, editor, *Multidimensional Systems Theory.* Dordrecht: Reidel, 1985.

[24] Turhan Ciftcibasi and Önder Yüksel. Sufficient or necessary conditions for model controllability and observability of Roesser's 2-D system model. *IEEE Trans. Automat. Control*, 28(4):527–529, 1983.

[25] D. L. Davis. A Correct Proof of Huang's Theorem on Stability. *IEEE Trans. Acoustics, Speech and Signal Processing*, 24(5):425–426, 1976.

[26] R. DeCarlo, J. Murray, and Raeks. Multivariate Nyquist Theory. *Int. J. Control*, 25:657–675, 1976.

[27] C. A. Desoer, R. Liu, J. Murray, and R. Saeks. Feedback systems design: the fractional representation approach to analysis and synthesis. *IEEE Trans. Automat. Control*, 25:399–412, 1980.

[28] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties.* Academic Press, 1975.

[29] D. E. Dudgeon and R. M. Mersereau. *Multidimensional Digital Signal Processing.* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1984.

[30] R. Eising. Realization and Stabilization of 2-D Systems. *IEEE Trans. Automat. Control*, 23(5):793–799, 1978.

[31] R. Eising. Controllability and observability of 2-D systems. *IEEE Trans. Automat. Control*, 24(1):132–133, 1979.

[32] R. Eising. State Space Realization and Inversion of 2-D Systems. *IEEE Trans. Circuits and Systems*, 27(7):612–619, 1980.

[33] E. Emre. Further Results on Skew-Prime Matrices For Linear Systems Over Rings. In *Proc.Conf. on Decision and Control*, pages 134–137, Orlando, FL., 1982.

[34] E. Emre and O. Huseyin. Relative primeness of multivariable polynomials. *IEEE Trans. Circuits and Systems*, 22:56–57, 1975.

[35] A. Fettweis and S. Basu. New Results on Stable Multidimensional Polynomials—Part I : Continuous Case. *IEEE Trans. Circuits and Systems*, 34(10):1221–1232, 1987.

[36] E. Fornasini. A note on output feedback stabilizability of multivariable 2D systems. *Syst. & Control Letters*, 10:45–50, 1988.

[37] E. Fornasini and G. Marchesini. Algebraic Realization Theory of Two-Dimensional Filters. In A. Ruberti and R. Mohler, editors, *Variable Structure Systems*, N.Y., 1975. Springer Verlag.

[38] E. Fornasini and G. Marchesini. State-Space Realization Theory of Two-Dimensional Filters. *IEEE Trans. Automat. Control*, 21:484–492, 1976.

[39] E. Fornasini and G. Marchesini. Two-Dimensional Filters: New Aspects of the Realization Theory. presented at Third Int. Joint Conf. on Patten Recongnition, Coronado, California, Nov. 1976.

[40] E. Fornasini and G. Marchesini. Doubly-Indexed Dynamical Systems: State-Space Models and Structural Properties. *Math. Systems Theory*, 12:59–72, 1978.

[41] E. Fornasini and G. Marchesini. On the Internal Stability of Two-Dimensional Filters. *IEEE Trans. Automat. Control*, 24(1):129–130, 1979.

[42] E. Fornasini and G. Marchesini. Stability analysis of 2D systems. *IEEE Trans. Circuits and Systems*, 27:1210–1217, 1980.

[43] A. Furukawa, T. Sasaki, and H. Kobayashi. Gröbner basis of a module over $K[x_1, \dots, x_n]$ and polynomial solutions of a system of linear equations. In B. W. Char, editor, *Proc. SYMSAC'86*, pages 222–224, Waterloo, Canada, 1986.

[44] Z. Geng, R. Carroll, and J. Xie. Two-dimensional model and algorithm analysis for a class of iterative learning control systems. *Int. J. Control*, 52(4):833–862, 1990.

[45] D. Goodman. An Alternate Proof of Huang's Stability Theorem. *IEEE Trans. Acoustics, Speech and Signal Processing*, 24:426–427, 1976.

[46] D. Goodman. Some stability properties of two-dimensional linear shift-invariant digital filters. *IEEE Trans. Circuits and Systems*, 24:201–208, 1977.

[47] J. P. Guiver. The Equation $Ax=b$ over the Ring $\mathbf{C}[z, w]$. In N. K. Bose, editor, *Multidimensional Systems Theory*. Dordrecht: Reidel., 1985.

[48] J. P. Guiver and N. K. Bose. Polynomial Matrix Primitive Factorization over Arbitrary Coefficient Field and Related Results. *IEEE Trans. Circuits and Systems*, 29:649–657, 1982.

[49] J. P. Guiver and N. K. Bose. Causal and Weakly Causal 2-D Filters with Applications in Stabilization. In N. K. Bose, editor, *Multidimensional Systems Theory*, page 52. Dordrecht: Reidel, 1985.

[50] T. S. Huang. Stability of two-dimensional recursive filters. *IEEE Trans. Audio Electroacoust*, 20:158–163, 1972.

[51] E. I. Jury. Stability of multidimensional scalar and matrix polynomials. *Proc. IEEE*, 66:1018–1047, 1978.

[52] T. Kaczorek. Dead-beat servo problem for 2-dimensional linear systems. *Int. J. Control*, 137(6):1349–1353, 1983.

[53] T. Kaczorek. Dead-beat servo problem for 2-D multivariable systems. In *Proc. 3rd Int. Conf. Syst. Eng.*, pages 524–529, Dayton, OH, 1984.

[54] T. Kaczorek. *Two-Dimensional Linear Systems*. Springer-Verlag, Berlin, 1985.

[55] T. Kailath. *Linear Systems*. Printice-Hall, 1980.

[56] E. W. Kamen. On the Relationship Between Zero Criteria for Two-Variable Polynomials and Asymptotic Stability of Delay Differential Equations. *IEEE Trans. Automat. Control*, 25:983–984, 1980.

[57] P. P. Khargonekar, T. T. Georgiou, and A. B. Özgüler. . *Linear Algebra Appl.*, 50:403–435, 1983.

[58] P. P. Khargonekar and E. D. Sontag. On the relation between stable matrix fraction factorizations and regulable realizations of linear systems over rings. *IEEE Trans. Automat. Control*, 27:627–638, 1982.

[59] J. Klamka. Controllability of M-dimensional linear systems. *Foundations of Control Engineering*, 8(2):65–74, 1983.

[60] S. Kodama and N. Suda. *Matrix Theory for System Control.* The Society of Instrument and Control Engineers of Japan, 1978. (in Japanese).

[61] J. Kollár. Sharp Effective Nullstellensatz. *Am. Math. Soc.*, 1:963–975, 1988.

[62] V. Kučera. *Discrete Linear Control: The Polynomial Equation Approach.* Wiley, Chichester, 1979.

[63] V. Kučera and M. Šebek. On deadbeat controllers. *IEEE Trans. Automat. Control*, 29:719–722, 1984.

[64] S. Y. Kung, B. Levy, M. Morf, and T. Kailath. New result in 2-D systems theory, Part II: 2-D State-Space Models—Realization and the Notions of Controllability, Observability, and Minimality. *Proc. IEEE*, 65(6):945–961, 1977.

[65] J. E. Kurek. Basic properties of q-dimensional linear digital systems. *Int. J. Control*, 42:119–128, 1985.

[66] J. E. Kurek. Reachability of a system described by the multidimensional Roesser model. *Int. J. Control*, 45:1559–1563, 1987.

[67] Y. S. Lai and C. T. Chen. Coprime fraction computation of 2D rational matrices. *IEEE Trans. Automat. Control*, 32:333–336, 1987.

[68] Z. Lin. Feedback stabilization of multivariable two-dimensional linear systems. *Int. J. Control*, 48:1301–1317, 1988.

[69] J. Lu and T. Yahagi. Pole-Zero Placement Adaptive Control for Nonminimum Phase Systems. In *Proc. of the 14th SICE Symp. on Dynamic System Theory*, pages 377–380, Okinawa, Japan, 1991.

[70] W. Marszalek. Two-dimensional state-space discrete models for hyperbolic partial differential equations. *Applied Mathematical Modelling*, 8:11–14, 1984.

[71] S. K. Mitra, A. D. Sagar, and N. A. Pendergrass. Realizations of Two-Dimensional Recursive Digital Filters. *IEEE Trans. Circuits and Systems*, 22(3):177–184, 1975.

[72] Y. Miyasato and Y. Oshima. A Design Method of Learning Control Systems. *Trans. SICE*, 23(6):576–583, 1987. (in Japanese).

[73] F. Mora and H. M. Möller. New constructive methods in classical ideal theory. *J. of Algebra*, 100(1):138–178, 1986.

[74] M. Morf, B. Levy, and S. Y. Kung. New result in 2-D systems theory, Part I: 2-D polynomial matrices, factorization and coprimeness. *Proc. IEEE*, 65(6):861–872, 1977.

[75] A. S. Morse. Ring Models for Delay-Differential Systems. *Automatica*, 12:529–531, 1976.

[76] J. Murray. Another proof and a sharpening of Huang's theorem. *IEEE Trans. Acoustics, Speech and Signal Processing*, 25:581–582, 1977.

[77] C. N. Nett, C. A. Jacobson, and M. J. Balas. A Connection Between State-Space and Doubly Coprime Fractional Representations. *IEEE Trans. Automat. Control*, 29:831–832, 1984.

[78] E. E. Newhall, S. U. H. Qureshi, and C. F. Simone. A Technique for Finding Approximate Inverse Systems and Its Application to Equalization. *IEEE Trans. on Communication Technology*, 19(6):1116–1127, 1971.

[79] V. R. Raman and R. Liu. A Constructive Algorithm for the Complete Set of Compensators for Two-Dimensional Feedback Systems Design. *IEEE Trans. Automat. Control*, 31:166–170, 1986.

[80] P. S. Reedy, S. R. Palacherla, and M. N. S. Swamy. Least Squares Inverse Polynomial and a Proof of Practical-BIBO Stability of n-D Digital Filters. *IEEE Trans. Circuits and Systems*, 37(12):1565–1567, 1990.

[81] R. P. Roesser. A Discrete State-Space Model for Linear Image Processing. *IEEE Trans. Automat. Control*, 20:1–10, 1975.

[82] E. Rogers and D. H. Owens. 2D Transfer-Functions Based Controller Design for Differential Linear Repetitive Processes. In V. Strejc, editor, *Proc. of the 2nd IFAC*

Workshop on System Structure and Control, pages 420–423, Prague, Czechoslovakia, 1992.

[83] E. Rogers and D. H. Owens. *Stability Analysis for Linear Repetitive Processes.* Springer Verlag, Berlin, 1992.

[84] R. Saeks and J. Murray. Feedback System Design: the tracking and disturbance rejection problems. *IEEE Trans. Automat. Control*, 26:203–217, 1981.

[85] S. Sakata. Synthesis of two-dimensional linear feedback shift registers and Gröbner bases. In L. Huguet and A. Poli, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes: Proc. of AAECC-5*, pages 394–407, Berlin, 1989. Springer Verlag.

[86] S. Sakata. Two-dimensional shift register synthesis and Gröbner bases for polynomial ideals over integer residue ring. In H. F. Mattson and J. T. Mora, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes: Proc. of AAECC-7*, France, Toulouse, 1989.

[87] S. Sakata. A Gröbner basis and a minimal polynomial set of a finite nD array. In S. Sakata, editor, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes: Proc. of AAECC-8*, Japan, Tokyo, 1990.

[88] S. Sakata. Finding a Gröbner basis of a module defined by a vector of nD arrays. In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes: Proc. of AAECC-9*, 1991.

[89] M. Šebek. On 2-D pole placement. *IEEE Trans. Automat. Control*, 30:820–822, 1985.

[90] M. Šebek. Asymptotic Tracking for 2D and Delay-differential Systems. *Automatica*, 24:711–713, 1988.

[91] M. Šebek. Controllability and Reconstructibility Conditions for 2-D Systems. *IEEE Trans. Automat. Control*, 33(5):496–499, 1988.

[92] M. Šebek. n-D polynomial matrix equations. *IEEE Trans. Automat. Control*, 33(5):499–502, 1988.

[93] M. Šebek. One More Counterexample in n-D Systems—Unimodular Versus Elementary Operations. *IEEE Trans. Automat. Control*, 33(5):502–503, 1988.

[94] M. Šebek. Two-sided equations and skew primeness for n-D polynomial matrices. *Syst. & Control Letters*, 12:331–337, 1989.

[95] S. Shankar and V. R. Sule. Algebraic Geometric Aspects of Feedback Stabilization. *SIAM J. Control and Optimization*, 30:11–30, 1992.

[96] J. L. Shanks, S. Treitel, and J. H. Justice. Stability and Synthesis of Two-Dimensional Recursive Filters. *IEEE Trans. Audio Electroacoust*, 20(2):115–128, 1972.

[97] M. G. Strintzis. Test of Stability of Multidimensional Filters. *IEEE Trans. Circuits and Systems*, 24(8):432–437, 1977.

[98] M. N. Swamy, L. M. Roytman, and E. I. Plotkin. On Stability Properties of Three- and Higher Dimensional Linear Shift-Invariant Digital Filters. *IEEE Trans. Circuits and Systems*, 32:888–891, 1985.

[99] N. J. Theordoru and S. G. Tzafestas. A canonical state-space model for 3-dimensional systems. *Int. J. Systems Sci.*, 15:1353–1379, 1984.

[100] S. G. Tzafestas, editor. *Multidimensional Systems—Techniques and Applications*, N.Y., 1986. Marcel Dekker.

[101] S. G. Tzafestas. State-Feedback Control of Three-Dimensional Systems. In S. G. Tzafestas, editor, *Multidimensional Systems—Techniques and Applications*, pages 161–232, N.Y., 1986. Marcel Dekker.

[102] S. G. Tzafestas and T. G. Pimenides. Exact-model-matching control of 3-D systems in state space. *Int. J. Systems Sci.*, 13:1171–1187, 1982.

[103] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* MIT Press, 1985.

[104] M. Vidyasagar, H. Schneider, and B. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. Automat. Control*, 27(4):880–894, 1982.

[105] B. Wall. Computation of Syzygies, Solution of Linear Systems over a Multivariate Polynomial Ring. Diploma Thesis, Johannes Kepler University, Linz, Austria., 1988.

[106] R. Whalley. Multidimensional systems and multirate sampling. *Appllied Mathematical Modelling*, 14:132–139, 1990.

[107] F. Winkler. Solution of equations I: Polynomial ideals and Gröbner bases. In D. V. Chudnovsky and R. D. Jenks, editors, *Computers in Mathematics*, pages 383–407. Marcel Dekker, 1990.

[108] W. A. Wolovich. Skew Prime Polynomial Matrices. *IEEE Trans. Automat. Control*, 23:880–887, 1978.

[109] W. A. Wolovich. Deadbeat error control of discrete multivariable systems. *Int. J. Control*, 37:567–580, 1983.

[110] L. Xu, K. Abe, and O. Saito. Characteristic Polynomial Assignment in 2-D Systems by Using Gröbner Bases. In *Proc. of International Session of SICE'89*, pages 1193–1196, Matsyama, Japan, 1989.

[111] L. Xu, T. Matsunaga, O. Saito, and K. Abe. Practical Tracking Problem of nD Systems and Its Applications. In *Proc. of SICE'92*, pages 14–15, Kumamoto, Japan, 1992. (in Japanese).

[112] L. Xu, O. Saito, and K. Abe. Tracking and Regulator Problems in 2-D Systems. In *Preprints of the 12th SICE Symp. on Dynamic System Theory*, pages 181–184, Toyohashi, Japan, 1989. (in Japanese).

[113] L. Xu, O. Saito, and K. Abe. Bilateral Polynomial Matrix Equations in Two Indeterminates. *Multidimensional Systems and Signal Processing*, 1:363–379, 1990.

[114] L. Xu, O. Saito, and K. Abe. A Note on Stabilization Algorithms for 2-D Systems. In *Proc. of the 13th SICE Symp. on Dynamical System Theory*, pages 333–336, Tokyo, Japan, 1991.

[115] L. Xu, O. Saito, and K. Abe. Stabilizability Test and Stabilizing Compensation for n-D Systems. In *Proc. of SICE'91*, pages 31–32, Yonezawa, Japan, 1991. (in Japanese).

[116] L. Xu, O. Saito, and K. Abe. Feedback Practical-Stabilization of nD Systems. *Trans. SICE*, 28(9):1103–1110, 1992. (in Japanese).

[117] L. Xu, O. Saito, and K. Abe. On 2D Bilateral Polynomial Matrix Equations. In H. Kimura and S. Kodama, editors, *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing II*, pages 177–182, Osaka, 1992.

[118] L. Xu, O. Saito, and K. Abe. On the Design of Practically-Stable nD Feedback Systems. In V. Strejc, editor, *Proc. of the 2nd IFAC Workshop on System Structure and Control*, pages 424–427, Prague, Czechoslovakia, 1992. also submitted to *Automatica*.

[119] L. Xu, O. Saito, and K. Abe. On the Practical-Asymptotic-Stability of nD Systems. In *Proc. of SICE'92*, pages 15–16, Kumamoto, Japan, 1992. (in Japanese).

[120] L. Xu, O. Saito, and K. Abe. Output Feedback Stabilizability and Stabilization Algorithms for 2D systems. submitted to *Multidimensional Systems and Signal Processing*, 1992.

[121] L. Xu, O. Saito, and K. Abe. Practical-Stabilization of nD Systems by State-Space Approach. In *Proc. of the 21th Symp. on Control Theory*, pages 191–196, Kariya, Japan, 1992. (in Japanese).

[122] L. Xu, O. Saito, and K. Abe. Practical Internal Stability of nD Discrete Systems. to be presented on the International Symposium MTNS-93, 1993.

[123] D. Youla and G. Gnavi. Notes on n-Dimensional System Theory. *IEEE Trans. Circuits and Systems*, 26:105–111, 1979.

[124] O. Zariski. *Commutative Algebra*, volume II. Springer-Verlag., 1960.

[125] S. Y. Zhang. Tests for regulability of 2-D systems. *Syst. & Control Letters*, 4:231–235, 1984.

Acknowledgements

This research was accomplished while the author was working as a doctoral candidate at Toyohashi University of Technology since April, 1988 to January, 1993. The author is fully aware of that this would have been impossible without the kind, continuous support and encouragement from his supervisors, colleagues, friends, and of course his family.

First of all, the author wishes to express his sincere gratitude to Professor Osami Saito, Professor Kenichi Abe (Toyohashi University of Technology), for kindling his interest to this exciting area and providing him a very free academic environment to carry out the research. More importantly, their kind trust, inspiring guidance and insisting encouragement, not only on the research but also on many other aspects, have contributed a lot to the progress of the research.

The author owes Professor Shojiro Sakata (Toyohashi University of Technology) a debt of gratitude, who has constantly supplied informative comments and valuable suggestions throughout the research, and has had a significant influence on the improvement of this thesis as a stringent and penetrating supervisor.

Many thanks are also due to Associate Professor Makoto Yoshizawa (Toyohashi University of Technology) for his helpful comments and suggestions in the examination of this thesis; and to Professor Yinling Zhang (Lanzhou University, China) for his kind guidance and constant encouragement.

The author would like to express his appreciation as well to Professor N. K. Bose (Pennsylvania State University) for his kind helps and encouragement; to Professor B. Buchberger, Dr. P. Paul, Dr. S. Shifter, Dr. H. Hong (all at Johannes Kepler University Linz, Austria) for affording the author the opportunity of visiting Research Institute for Symbolic Computation and/or helpful and impressive discussions during the author's short stay there; to Professor G. Marchesini, Professor E. Fornasini (both at University of Padua, Italy), Dr. M. Šebek (Czechoslovak? Academy of Science) for some valuable discussions and for their kind invitations which supplied the author opportunities of presenting some results of this research at international conferences; and to Professor T. Sasaki (Tsukuba University, Japan) for his instructive and encourage discussions and offering of original references.

Concerning the completion of this research, the author is very grateful for the valuable work of Wataru Ushio, Takatoshi Ikeda and Toshio Matsunaga, who implemented partly the results of this research on computer and helped a lot for the computation and simulation of examples during their study at Toyohashi University of Technology as Master students. Also, the author is indebted to B. Wall, who is now working towards Ph.D degree at Johannes Kepler University Linz, for his friendly and valuable help in computing the example of Chapter 4 by using their algebraic computing system.

Special thanks are owing to State Education Commission of the People's Republic of China and Lanzhou University for affording the author the opportunity of studying in Japan and the financial support, and to the Ministry of Education of Japan for the financial support and the agreeable accommodations.

The author gratefully acknowledges his many Japanese and Chinese friends whose kindness, friendship, hospitality and assistance made the stay of the author along with his wife and daughter in Toyohashi so enjoyable and unforgettable. However, the author "cannot name them all because he can't be sure that he would not miss one or two".

Last, but never least, the author would like to give his deep gratitude to his wife and daughter, his parents, his wife's mother and late father and all the other members of his family, who were always a great source of support and encouragement during the research that took so much time of the author away from them.