# On Associative Memory Neural Network

January, 1994

DOCTOR OF ENGINEERING

Xin-Min HUANG

TOYOHASHI UNIVERSITY OF TECHNOLOGY

# On Associative Memory Neural Network

January, 1994

DOCTOR OF ENGINEERING

Xin-Min HUANG

TOYOHASHI UNIVERSITY OF TECHNOLOGY

# 連想記憶ニューラルネットワーク
# に関する研究

## 要　旨

近年、多数の簡単な処理ユニットを持つ高度な並列情報処理システム、あるいは、人工的なニューラルネットワークに関する研究が盛んになってきている。このシステムを用いることにより、画像処理や最適化など従来のコンピュータが実行困難である問題を有効に解決することが期待できる。

本研究は、連想記憶ニューラルネットワークモデルに関する検討を行った。連想記憶ニューラルネットワークが人工的なニューラルネットワークの一つである。本研究の目的は次の通りである。

- 従来の連想記憶ニューラルネットワークモデルの基本特性を解析し、その中に残った問題及びこの問題を解決する方法を検討する。
- 一般化した連想記憶ニューラルネットワークモデル及びこのモデルにおける重み行列を有効に求める方法を提案する。このモデルは無条件総合安定であり、更に大きな記憶容量と効率的な動的特性を持つことが期待される。

この目的に対して、まず、我々は人工的なニューラルネットワークの定義及び従来の連想記憶ニューラルネットワークモデルを検討した。その上で、一つの半直交連想記憶ニューラルネットワーク（SAM）モデルを提案した。また、統計神経動力学を用いて、SAMモデルの基本特性を調べた。

本論文は六章から構成されている。第一章では一つの緒論を与える。第二章では、Hopfieldモデルと BAMモデルにおける問題点を検討して、一般化した連想記憶ニューラルネットワークモデルを提案した。Hopfieldモデルと BAMモデルは従来の連想記憶ニューラルネットワークの代表的なモデルと考えられる。本モデルは、Hopfieldモデルと BAMモデルの組合せであると位置づけられる。しかし、BAMモデルに存在した初期入力方向による連想の出力に多義性があるという問題点は本モデルにはない。更に、本モデルは無条件総合安定である。第三章では連想記憶モデルにおける連想の相似確率を定義して、この確率に関する基本命題を与えた。また、統計神経動力学を利用して、$N$個のニューロンを有するHopfieldモデルの記憶容量が$2N/\pi$以下であることを証明した。第四章では一つの動的閾値を連想記憶モデルに導入することにより、このモデルにおける連想の出力と学習パターン間の相関性を改善する。最適な動的閾値を持つHopfieldモデルをODAMモデルと呼ぶ。ODAMモデルの動的特性は従来のHopfieldモデルより効率的であることを理論解析及びコンピュータシミュレーションによって確認した。更に、第五章では一般化した連想記憶モデルにおける効率的な重み行列を求める半直交学習法を提案した。半直交学習法を用いて求めた重み行列を持つ本連想記憶モデルをSAMモデルと呼ぶ。我々はSAMモデルの動的特性を検討し、このモデルの収束性を解明した。SAMモデルにおける一つの結合重みに当たる情報蓄積量は1/4(bits/weight)以上であり、$N$個のニューロンを有するSAMモデルの記憶容量は$N/2\ln\ln N$である。第六章では本研究の主な結果のまとめと本モデルに残された問題及び展望について述べた。



# On Associative Memory Neural Network

## Abstract

This thesis considers the Associative Memory (AM) neural network with feedback—an important type of artificial neural network. The principle objective of this research is:

**First:** to analyze the fundamental properties of the conventional AM models, investigate the problems still remaining in them and discuss how to solve or improve these problems.

**Second:** to propose a generalized AM model and a training algorithm for determining its connection weights. This model is expected to be unconditionally globally stable, and have a larger memory capacity and better dynamic properties.

We begin with the discussion of the original definition of artificial neural network and the limitations of conventional AM models and then propose a semi-orthogonally associative memory model corresponding to the principle objective. The semi-orthogonally associative memory model is simply called SAM model. By means of statistical neurodynamics and a Lyapunov function, the fundamental properties of SAM model are investigated.

This thesis consists of 6 chapters. Chapter 1 is introductory in nature. In Chapter 2, the limitations of Hopfield model and BAM model, representatives of the conventional AM models, are discussed and then a generalized form of AM model with feedback is proposed. The generalized AM model can be regarded as a combination of Hopfield model and BAM model. This model is unconditionally globally stable and has no the problem which the recalling outputs depend on the directions that the initial inputs are entered like in BAM model. Chapter 3 defines a statistical variable, called similar

probability in the recalling processes of AM model, and gives the fundamental lemmas and important results in terms of statistical neurodynamics. Further, the fundamental properties of conventional AM models are discussed with the similar probability. It is proved that the memory capacity of Hopfield model with $N$ neurons is less than $2N/\pi$. In Chapter 4, we introduce a dynamic threshold to AM model to reduce the dependence of outputs in the recalling processes on all the learned patterns. By means of statistical neurodynamics, the optimum dynamic threshold in Hopfield model is derived. Theoretical analysis and simulation experiments confirmed that Hopfield model with the optimum dynamic threshold, called ODAM model, has better dynamic properties. Chapter 5 gives a semi-orthogonally training algorithm for determining the connection weights in the generalized AM model. The model with the connection weights decided by this algorithm is called SAM model. We explicate the convergence of SAM model and give its convergence criteria. Its information storage capacity per weight is larger than $1/4$ (bits/weight) and its memory capacity is $N/2\ln\ln N$, where $N$ is the number of neurons. The limitations of dynamic threshold in SAM model are investigated as well. Finally, Chapter 6 gives a brief summary of the main results of this thesis and some remarks on possibilities and problems for the further research.

# Contents

# Chapter 1

# Introduction

## 1.1 Background and Objectives of this Research

The area of artificial neural network has received much attention recently. Artificial neural network is a special computing architecture and the discovery of its latent abilities are only just begun. The completely new computing architecture suggested by the structure of biological nervous system is different from that of the conventional computer architecture, which is sequential and programmable. This new computing architecture is highly parallel and does not need programming. It is expected to be an effective instrument to solve many problems such as pattern recognition and classification, optimization, etc. These problems are difficult for electronic computers.

An artificial neural network is a huge system which is composed of a vast number of very simple processing elements and connection weights between the elements, sometimes euphemistically called "neurons" and "synapsis" between the neurons, respectively. These wonderfully simple processing elements are closely connected to each other by the connection weights and compute their outputs by a threshold function of their postsynaptic potentials for any inputs. This is similar to the fundamental structure of human nervous system, although some details are considerably different. Artificial neural networks can be expected to bring about drastic advancements to present day electronic computers, and can receive the challenge of better understanding of the working of human brain. It is one of the biggest challenges of scientific research, at present.

### 1.1.1 Background

Since the late middle ages, the human central nervous system has been studied. But, some of its detailed structure began to be unraveled only a century ago. And the study of artificial neural networks, as depicted in Figure 1-1, was begun only in 40s years of this century. Around the same time, the first electronic computer was born. However, the development of artificial neural network is far removed from that of the computer.

In 1943, Warren McCulloch and Walter Pitts[133] proposed a general theory of information processing based on the binary switching network or decision elements, which are somewhat euphemistically called "neurons". The $i$th neuron $x_i$ $(i = 1, \cdots, N)$ can be in one of two states: $x_i = 0(\text{off})$ or $x_i = 1(\text{on})$. In order to simulate the finite regenerative period of real neurons, changes in the states of this network are supposed to occur in discrete time steps $t = 0, 1, \cdots$. The new state of a certain neuron is determined by the influence from all other neurons, and expressed by a linear combination of the inputs:

$$h_i(t) = \sum_j w_{ij} x_i(t). \tag{1.1}$$

Here $w_{ij}$ represents the connection weight from the $j$th neuron to the $i$th, while $h_i(t)$ models the total postsynaptic potential at neuron $i$ caused by all other neurons. The dynamic properties of this network are determined by the functional relation between $h_i(t)$ and $x_i(t+1)$. In general, $x_i(t+1) = 1(\text{on})$ if $h_i(t)$ exceeds a certain threshold $\vartheta_i$, and $x_i(t+1) = 0(\text{off})$ otherwise. In other words, the state evolution of neuron $i$ is governed by the law



| Year | Author | Terms |
|---|---|---|
| 1943 | McCulloch et al | artificial neural network |
| 1949 | Hebb | learning law |
| 50s-60s | Minsky et al | Perceptron |
| 1969 | Minsky & Papert | pointed out problems in single-layer Perceptron (exclusive-OR gate) |
| 70s | Kohonen et al, Anderson et al | AM model |
| 1982 | Hopfield | AM model with feedback (energy function) |
| 1988 | Kosko | BAM model |

**Figure 1-1:** A history of research on artificial neural networks

$$x_i(t+1) = f(h_i(t) - \vartheta_i) \qquad (t = 0, 1, \cdots) \tag{1.2}$$

where $f(\xi)$ is the unit step function: $f(\xi) = 0$ $(\xi < 0)$ and $f(\xi) = 1$ $(\xi > 0)$. That is, the network is defined on the set $\mathrm{V}^N = \{0, 1\}^N$, which is called the state space of this network.

Generally, the function $f$ can be an arbitrary sigmoid function. In this thesis, the function $f$ is considered as the sign function, namely, $f(\xi) = -1$ ($\xi < 0$) and $f(\xi) = 1$ ($\xi \geq 0$). That is, the state space is set $\mathbf{U}^N = \{-1, 1\}^N$. These models are equivalent mutually whether the state space is set $\mathbf{V}^N$ or $\mathbf{U}^N$.

McCulloch and Pitts[133] showed that such network can, in principle, carry out any imaginable computation, similar to a programmable, digital computer. The remaining problem is how to decide the connection weights when the network is designed so that a specific task can be performed by the machine. Here the word "task" is in a generalized sense: it can mean any task requiring digital or analog information processing, such as the recognition of specific optical or acoustical patterns. This problem was discussed by Eduardo Caianiello[33], who gave a "training" algorithm that would allow us to construct the connection weights in the artificial neural network. This algorithm incorporates in a simple way the basic principle of Hebbian[77] learning rule.

Around 1960 Frank Rosenblatt et al[164], studied a type of neural network model called perceptron, which consists of two separate layers of neurons representing the input and output layer, respectively. They introduced an iterative algorithm for constructing the connection weights such that an input is transformed to a special output pattern which is termed the desired pattern on this initial input, and even succeeded in proving its convergence. However, Marvin Minsky and Seymore Papert[140] pointed out a few years later that this proof applies only to those problems which can, in principle, be solved by a perceptron. What made matters worse was that they showed the exclusive-OR (XOR) gate can not be solved by any such two-layered perceptrons. The logic XOR gate is a standard problem easily solved in computer design, and thus the results of Minsky and Papert represented a severe blow to the perceptron concept. In fact, the XOR problem can be easily solved by a three-layered perceptron[167], but, at that time, no practical training algorithm



for determining the connection weights in such generalized perceptrons was known. The multilayered, feed-forward artificial neural network—perceptron was revived by the discovery of an efficient training algorithm for constructing the connection weights in it. This algorithm, now known as error back propagation, was initially suggested by Werbos[196].

Differing from the layered type of artificial neural networks, associative memory model is a connected type of artificial neural network. This network was proposed by several researching groups[8],[18],[103] at the beginning of the 1970s and has been attracting considerable attention. In particular, in 1982, J.Hopfield[81] demonstrated that the associative memory model with feedback is globally stable under some conditions, and in 1985, J.Hopfield and D.Tank[84] addressed using this model as an effective tool to solve the traveling salesman problem, which may be an NP-complete problem. Usually, the conventional associative memory model with feedback is simply called Hopfield model, or associative memory (AM) model.

In the last ten years, a large number of research results on the associative memory model have been published, and most of the fundamental properties of this model have been revealed. Applications and all optical or optoelectronic implementations[48],[49] on this model have been widely discussed. A great number of improved versions of the conventional associative memory model have also been proposed.

J.Hopfield[81],[82] introduced a Lyapunov function to Hopfield model. Sometimes, the Lyapunov function is also called state energy function. By explaining the decrease of Lyapunov function with time in the recalling processes, he demonstrated that the model is globally stable under the conditions: the state evolutions of neurons are asynchronous, the connection weight matrix is symmetric and its diagonal elements are not less than zero. An associative memory model is globally stable if the output series in the recalling processes is convergent for any initial input. In 1985, making use of this method, Gloes-Chace et al[61], made a detailed discussion on the global stability of Hopfield model. There is no doubt that it opened a new and powerful way to analyze the global stability of associative memory model by means of a Lyapunov function.

However, the condition which the state evolutions of neurons are asynchronous implies that neurons are updated sequentially only one at a time. It holds the essential conceptual disadvantage that the basic feature of neural networks, namely the simultaneous operation of a large number of parallel elements, is given up. Neurons in the human brain surely do not operate sequentially, this being precisely the reason for the brain's superiority in complex tasks to even the fastest existing electronic computer. In addition, the restrictions on the connection weight matrix are a disadvantage to discover the efficaciously training algorithm for constructing the connection weights. It is hard to imagine that the connection weights from the $i$th neuron to $j$th and from the $j$th to $i$th in the human nervous system must be equal to one another.

Y.Abu-Mostafa et al[1] discussed the information capacity of associative memory model based on threshold function theory and proved that, for a given model with $N$ neurons, its memory capacity is not larger than $N$. In other words, neuron number $N$ is an upper bound of memory capacity of associative memory model. By the simulation experiments, J.Hopfield[81] estimated that the memory capacity of Hopfield model is not larger than 0.15-times its neuron number. D.Amit et al[15],[16] did a statistical analysis on this model based on the equilibrium theory. R.Mceliece et al[134] utilized the techniques of coding theory to study the memory capacity of Hopfield model. In order that the most original learned patterns are exactly recoverable by Hopfield model with $N$ neurons, they demonstrated that, if learned patterns are random in this model, the maximum asymptotic value of the learned patterns number $m$ is $N/2\ln N$. That is, the memory capacity of Hopfield



model is

$$m_c = \frac{N}{2\ln N}, \tag{1.3}$$

as the neuron number $N \longrightarrow \infty$.

By means of statistical neurodynamics, S.Amari[14] et al analyzed the dynamic behavior in the recalling processes of Hopfield model and obtained about the same results on the memory capacity of this model. They also elucidated that there exist the strange dynamic behavior in Hopfield model, which is due to the strange shapes of equilibrium state attracting basins of this model. However, they did not discuss how to improve the strange dynamic behavior. Statistical neurodynamics has been applied to analyze the macroscopic dynamic properties of associative memory model by many researchers. For example, E.Harth et al[74], analyzed multistable characteristics of the associative memory model and S.Amari[9],[12] analyzed their oscillatory behaviors.

S.Aiyer et al[4] utilized subspace geometry theory to analyze the distribution of equilibrium states in the state space of associative memory model. He proved that spurious equilibrium states can occur at any corner of the hypercube which is on, or near, the subspace spanned by the learned patterns in the state space on which the model is defined. But, in the recalling processes of associative memory model, it has not been studied how to discriminate whether an equilibrium state is spurious or non-spurious.

In 1987, B.Kosko[110],[111],[116] proposed an associative memory model which consists of two neuron fields and where the recalling processes are bidirectional. He referred to this model as Bidirectional Associative Memory (BAM) model. In this model, learned patterns are respectively stored at these two neuron fields and the recalling processes can start from any one of the two neuron fields. The respective learned patterns stored at the two neuron fields can be completely different. Like in Hopfield model, by introducing a Lyapunov function, B.Kosko proved that BAM model is unconditionally globally stable. But, since there are two neuron fields in this model, in order to guarantee that the desired patterns can be correctly retrieved for any initial input, it must be known that the recalling processes ought to be begun from which one of the two neuron fields before they are done. In other words, for any initial input, before the recalling processes are begun, we must have known that the desired pattern on this initial input is stored at which one of the two neuron fields. This requirement is unreasonable and can not be satisfied in terms of any associative memory models.

In the human nervous system, if it is not getting any trouble, before the recalling processes are begun, requiring the teaching signal about the desired patterns is not imaginable, because the essential task of associative memory is just retrieving the desired pattern for any initial input. Although there exists this problem in BAM model, it is no doubt that this model suggests a good direction for proposing the unconditionally globally stable associative memory model. Hopfield model and BAM model can be considered as representatives of the conventional associative memory models.

### 1.1.2 Objectives of this Research

As mentioned previously, associative memory model—a type of important artificial neural network model, has been widely researched. However, up to now, its fundamental properties have not been properly explained. The phenomena that its equilibrium state attracting basins have strange shapes[14] have not been improved and its memory capacity need be greatly enlarged. Further, the ambiguity of the recalling outputs in BAM model must be solved as well.

In other words, it is expected to propose an unconditionally globally stable associative memory model. In this model, any problems that are



difficult to explain by the essential structure of human nervous system are not remained as possible, such as the problems in Hopfield model and BAM model. Corresponding to these expectations, the main objectives of this research are:

**First:** to thoroughly analyze the fundamental properties of conventional associative memory models, discuss the problems still remaining in them and investigate the methods for solving or improving the problems and enlarging their memory capacity.

**Second:** to propose an unconditionally globally stable associative memory model and a more efficient training algorithm for constructing its connection weight matrix. This model is expected to have much larger memory capacity, much better dynamic behavior than the conventional associative memory models. In this model, the phenomena that the equilibrium state attracting basins have strange shapes ought to not occur or are greatly improved at least, and the problems still remaining in BAM model must not exist.

In addition, we desire that this model can be easily extended to multidimensional form for multidimensional information processing, in particular, 2-dimensional case. And it should also be amenable to optical or opto-electronic implementations. Further, in a sense, this model is expected to have the ability to discriminate whether an equilibrium state is spurious or non-spurious for any initial input if there are spurious equilibrium states. In this thesis, the discrete associative memory model with feedback is the one mainly considered.

## 1.2 Main Results of this Research

In order to accomplish the main objectives of this research, we have made many theoretical analyses and simulation experiments with computer. By means of the statistical neurodynamics, the fundamental properties of the conventional associative memory models are analyzed in detail. After studying the original definition of artificial neural network, we propose a generalized associative memory model with feedback. We discuss the dynamic behavior of this model and demonstrate that it is unconditionally globally stable. The main results of this research are shown in following.

Firstly, the relations among the conventional associative memory models and the limitations of these models are discussed[89],[92]. Based on these, a generalized form of associative memory model is given. This model is made up of $N$ neurons and $n$ characteristic sites which are completely independent of the neurons. Here, we refer to the characteristic site number $n$ as characteristic parameter of this model. In this generalized model, the neurons are connected to each other through these characteristic sites. This is different from Hopfield model, in which the neurons are directly connected to each other. It is also different from BAM model, in which the neurons are divided into two neuron fields and the neurons within a neuron field are not directly connected to each other but only indirectly connected through the other neuron field. Like in the conventional associative memory models, we also introduce a Lyapunov function to this model, and demonstrate that this model is unconditionally globally stable by this function.

Secondly, we define a statistical variable called similar probability in the recalling processes of associative memory model. The similar probability is a fundamental statistical variable in terms of statistical neurodynamics. It indicates the statistical dynamic behavior of associative memory model. By discussing the probability in Hopfield model, we deduce a statistical upper-



bound of the memory capacity of Hopfield model[90]. That is, in a Hopfield model with $N$ neurons, the memory capacity is

$$m_c < \frac{2}{\pi} N.$$

Further, we introduce a dynamic threshold to associative memory model for improving the dependence of recalling outputs on all learned patterns and the strange shapes of equilibrium state attracting basins. By means of statistical analysis, we discuss the methods to determine the optimum dynamic threshold in Hopfield model. We refer to the Hopfield model with the optimum dynamic threshold as ODAM model. The theoretical analysis demonstrates that this model is of better dynamic behavior and larger memory capacity. These theoretical results are confirmed by the simulation experiments.

Thirdly, based on the Hebbian[77] learning principle, we proposed a semi-orthogonal training algorithm for constructing the connection weight matrix of generalized associative memory model. That is, we choose the orthogonal patterns in set $\{-1, 1\}^n$ as the characteristic patterns in $n$ characteristic sites of this model. The orthogonal patterns are completely independent of concrete learned patterns. The generalized associative memory model, in which the connection weight matrix are constructed by the semi-orthogonal training algorithm, is referred to as Semi-Orthogonally Associative Memory (SAM) model. We also estimate the similar probability in SAM model. By means of this probability we demonstrate that the memory capacity of a SAM model with $N$ neurons is

$$m_c = \frac{N}{2 \ln \ln N}. \tag{1.4}$$

Furthermore, we discuss how to determine the characteristic parameter $n$ for a given SAM model with $N$ neurons. According to the theoretical analysis, the optimum characteristic parameter is $n = 2^r$ and

$$\tau = [\log_2 2m_c] + 1.$$

where $m_c$ is the memory capacity of SAM model shown in equation (1.4).

The information storage capacity per connection weight in associative memory model is defined as follows.

$$S_c = \frac{\text{bit number of stored information}}{\text{number of connection weights}} \quad \text{(bits/weight)} \qquad (1.5)$$

This is an important parameter for associative memory model. Like the memory capacity of a model, it also indicates the information storage capacity of a model. But, for an associative memory model, in general, $S_c$ is not equal to the memory capacity per neuron except that its connection weight matrix is square. In a SAM model with $N$ neurons, the information storage capacity per connection weight satisfies the following inequality when the characteristic parameter is optimum.

$$S_c > \frac{1}{4} \quad \text{(bits/weight)}.$$

Finally, we address the dynamic threshold in SAM model and analyze its limitations.

## 1.3 The Structure of this Thesis

This thesis consists of 6 chapters and its structure is shown in Figure 1-2. After giving an introduction in this chapter, the limitations of Hopfield model and BAM model are discussed and then the generalized associative memory model is proposed in Chapter 2. We first give examples[89] to show that Hopfield model is not globally stable if any one of the conditions give by



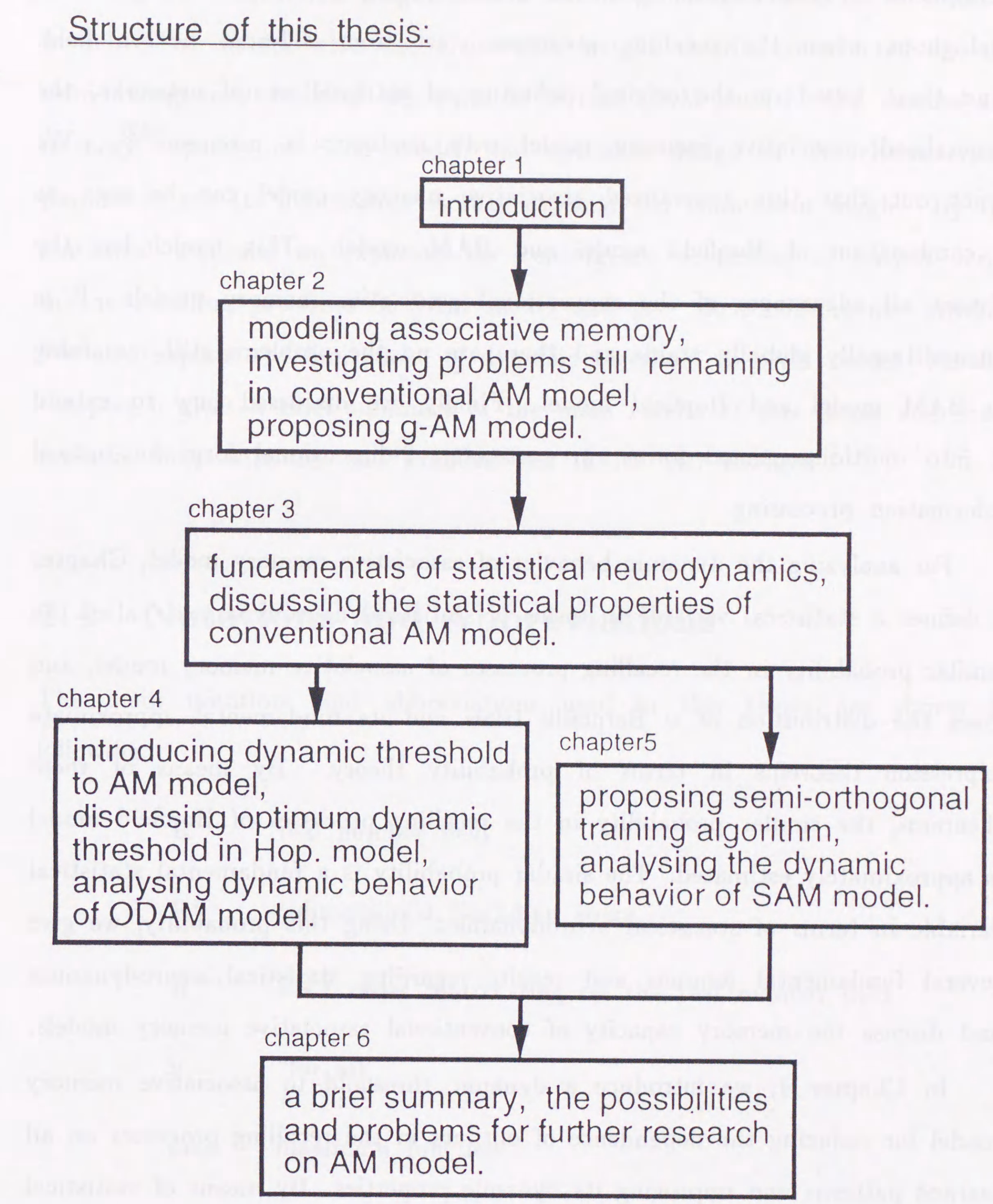



Figure 1-2: Structure of this thesis

J.Hopfield[81] is not true, and the recalling outputs of BAM model are ambiguous when the recalling processes start from different neuron field. And then, based on the original definition of artificial neural networks, the generalized associative memory model with feedback is proposed[92]. We point out that this generalized associative memory model can be seen as a combination of Hopfield model and BAM model. This model has the almost all advantages of the conventional associative memory models. It is unconditionally globally stable and there are no the problems still remaining in BAM model and Hopfield model. Finally, we discussed how to extend it into multidimensional forms, in particular, 2-dimensional form for optical information processing.

For analyzing the dynamic behavior of associative memory model, Chapter 3 defines a statistical variable in terms of statistical neurodynamics, called the similar probability in the recalling processes of associative memory model, and gives the distribution of $v$ Bernoulli trials and its fundamental approximate expression theorems in terms of probability theory. By means of these theorems, the similar probability in the recalling processes of Hopfield model is approximately estimated. The similar probability is a fundamental statistical variable in terms of statistical neurodynamics. Using this probability, we give several fundamental lemmas and results regarding statistical neurodynamics and discuss the memory capacity of conventional associative memory models.

In Chapter 4, we introduce a dynamic threshold to associative memory model for reducing the dependence of outputs in the recalling processes on all learned patterns and improving its dynamic properties. By means of statistical neurodynamics, the optimum dynamic threshold in Hopfield model is derived. The Hopfield model with the optimum dynamic threshold is simply called ODAM model[90]. Theoretical analysis and simulation experiments confirmed that this model has better dynamic properties and larger memory capacity.

Chapter 5 proposes the semi-orthogonally training algorithm for con-



structing the connection weight matrix in the generalized associative memory model. The model with the connection weight matrix constructed by the semi-orthogonally training algorithm is termed SAM model. We discuss the memory capacity of SAM model, the optimum design of the characteristic parameter and its information storage capacity per connection weight. By the statistical analysis, we explicate the convergence properties of output series in the recalling processes of SAM model and give their convergence criteria. We also explain the limitations of dynamic threshold in SAM model. Finally, Chapter 6 gives a brief summary of the main results of this thesis and some remarks on possibilities and problems for the further research.

## 1.4 Notations and Abbreviations

The main notations and abbreviations used in this thesis are shown in following.

| | |
|---|---|
| $\mathrm{R}$ | real number field |
| $\mathrm{R}^t$ | $t$-dimensional Euclidian space |
| $\mathrm{R}^{s\times t}$ | $s \times t$ order matrix ring on the real number field |
| $\forall$ | "for all" |
| max | maximum function |
| min | minimum function |
| $[\cdot]$ | integral function |
| $\|\cdot\|$ | absolute value of real number |
| $\|\cdot\|_s$ | number of elements in a set |

$\|\cdot\|_1$ norm 1 of vector in space $\mathbf{R}^t$

$A^T$ transpose of matrix $A$

$A \subset B$ $A$ is a subset of set $B$

$A \cap B$ intersection of sets $A$ and $B$

$A \cup B$ union of sets $A$ and $B$

$A \backslash B$ complement of set $B$ in set $A$

$A^c$ complement of set $A$

$\mathbf{U}^t = \{-1, 1\}^t$ $t$-dimensional polar binary vector set

$\mathbf{V}^t = \{0, 1\}^t$ $t$-dimensional binary vector set

$D(\cdot, \cdot)$ Standardized Effective Hamming Distance on set $\mathbf{U}^t$, which is defined as follows. For any $X, Y \in \mathbf{U}^t$ the standardized effective Hamming distance between $X$ and $Y$ is

$$D(X, Y) = \min\left\{\frac{\|X - Y\|_1}{2t}, \frac{\|X + Y\|_1}{2t}\right\};$$

$O(A, r)$ spherical neighborhood of point $A$ with radius $r$

$O(r)$ spherical neighborhood of a point with radius $r$

$P\{A\}$ probability of event $A$ in probability space

$\mathbf{E}(\chi)$ expectation of random variable $\chi$

$\mathbf{D}(\chi)$ variance of random variable $\chi$

$N$ number of neurons in a neural network

$m$ number of learned patterns in a neural network



$n$ characteristic parameter of SAM model

$m_c$ memory capacity of an associative memory model

$S_c$ information storage capacity per connection weight in an associative memory model

$p(t)$ similar probability between the output at time $t$ and the desired pattern on its initial input in the recalling processes of associative memory model

$\mathbf{W}$ connection weight matrix in associative memory model

$A^{(\xi)}$ learned pattern in a neural network ($A^{(\xi)} \in \mathbf{U}^N$, $1 \le \xi \le m$)

$O^{(\xi)}$ orthogonal vector in space $\mathbf{R}^t$ ($1 \le \xi \le m$)

equilibrium state: minimum point of a Lyapunov function in associative memory model

spurious equilibrium state: it is an equilibrium state but not a learned pattern in associative memory model

non-spurious equilibrium state: it is an equilibrium state and a learned pattern in associative memory model

desired pattern on initial input $A$: it is a learned pattern and the distance from it to $A$ is minimum on all learned patterns

# Chapter 2

# Associative Memory Models with Feedback

## 2.1 Models of Artificial Neural Network

Neural network is a system made up of a great number of neurons. Each neuron is a wonderfully simple information processing element which has multi-inputs and one-output. For a certain neuron $i$ with $N$ inputs $x_1(t), \cdots, x_N(t)$, the postsynaptic potential at this neuron $h_i(t)$ satisfies the following differential equation.

$$\tau \frac{dh_i(t)}{dt} = -h_i(t) + F_{\mathbf{W}}(x_1(t), \cdots, x_N(t)) \tag{2.1}$$

where $\mathbf{W}$ is the connection weight matrix in this network, $F_{\mathbf{W}}$ is a multivariate real function which depends on the connection weights, and $\tau$ is a time proportion constant, respectively.

Corresponding to the postsynaptic potential $h_i(t)$, the output $x_i(t)$ is determined by a sigmoid function of the potential $h_i(t)$, namely

$$x_i(t) = Sig(h_i(t) - \vartheta_i(t)) \tag{2.2}$$

where $Sig(\cdot)$ is a sigmoid function from the real number field $\mathbf{R}$ onto the interval $[-1, 1]$, and $\vartheta_i(t)$ is a threshold in neuron $i$. In general, the threshold is a function of its order and time $t$ in the state evolution processes. Figure 2-1 shows the neuron model which receives input signals $x_j(t)$ and has connection weights $w_{ij}$ $(j = 1, \cdots, N)$.

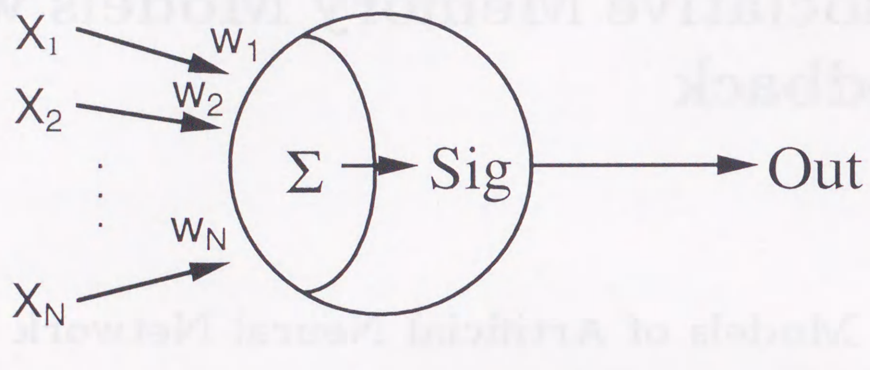



**Figure 2-1:** A neuron model

Then, for all neurons in the network, the state evolutions can be given in a vector form as follows.

$$\tau\frac{d\mathbf{H}(t)}{dt} = -\mathbf{H}(t) + F_{\mathbf{W}}(\mathbf{X}(t)) \tag{2.1$'$}$$

$$\mathbf{X}(t) = Sig(\mathbf{H}(t) - \vartheta(t)) \tag{2.2$'$}$$

The neural network, defined by equations (2.1) and (2.2), is a continuous neural network. Here, if we let the time proportion constant $\tau = 1$ in equation (2.1), function $Sgn(\cdot)$ be the sign function with threshold $\vartheta$, namely,

$$Sgn(x) = \begin{cases} 1 & x \geq \vartheta \\ -1 & x < \vartheta \end{cases} \tag{2.3}$$



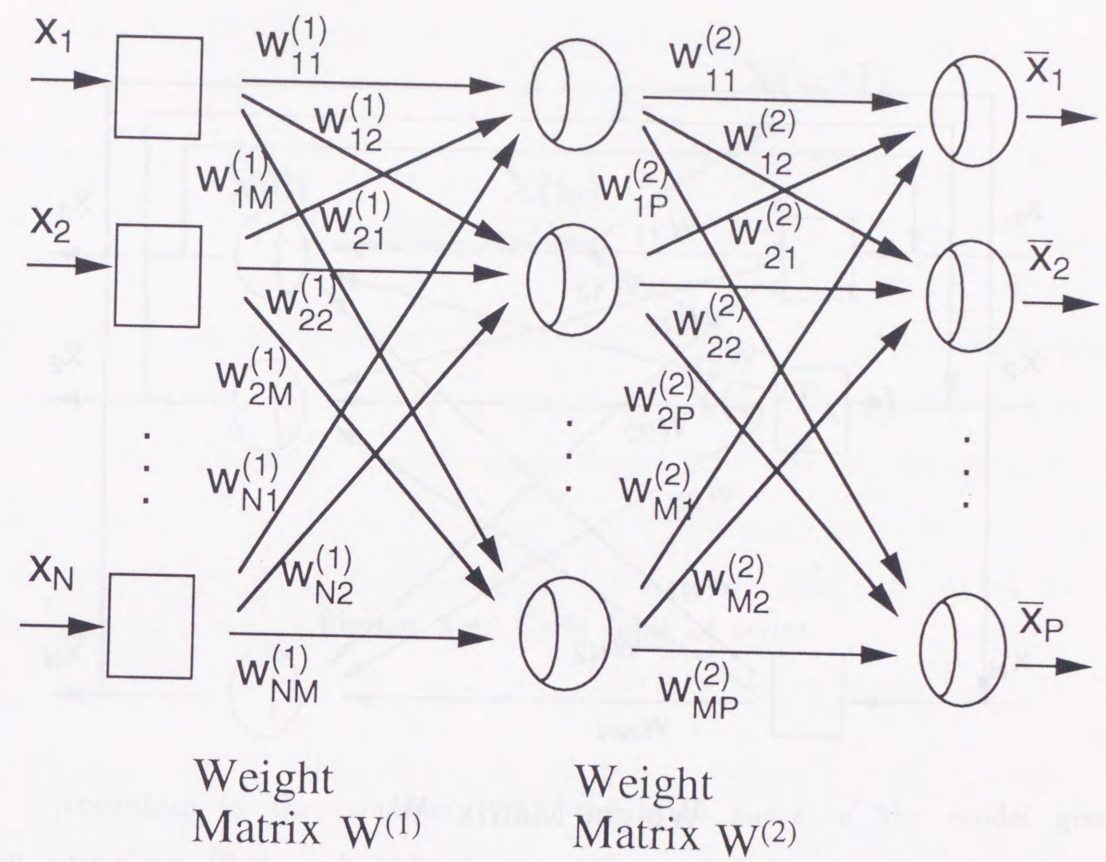



**Figure 2-2:** Two-layered feed-forward neural network

instead of the sigmoid function $Sig(\cdot)$ in equation (2.2), and consider the time to be discrete, then we can obtain the discrete neural network, in which the state evolutions of neurons are governed by the following law.

$$\mathbf{H}(t+1) \;=\; F_{\mathbf{W}}(\mathbf{X}(t)) \tag{2.4}$$

$$\mathbf{X}(t+1) \;=\; Sgn(\mathbf{H}(t+1) - \vartheta(t+1)) \tag{2.5}$$

Here, if the state evolutions of neurons are layer by layer feed-forward, the network is called layered feed-forward neural network. Figure 2-2 shows a two-layered feed-forward neural network. And if the state evolutions of

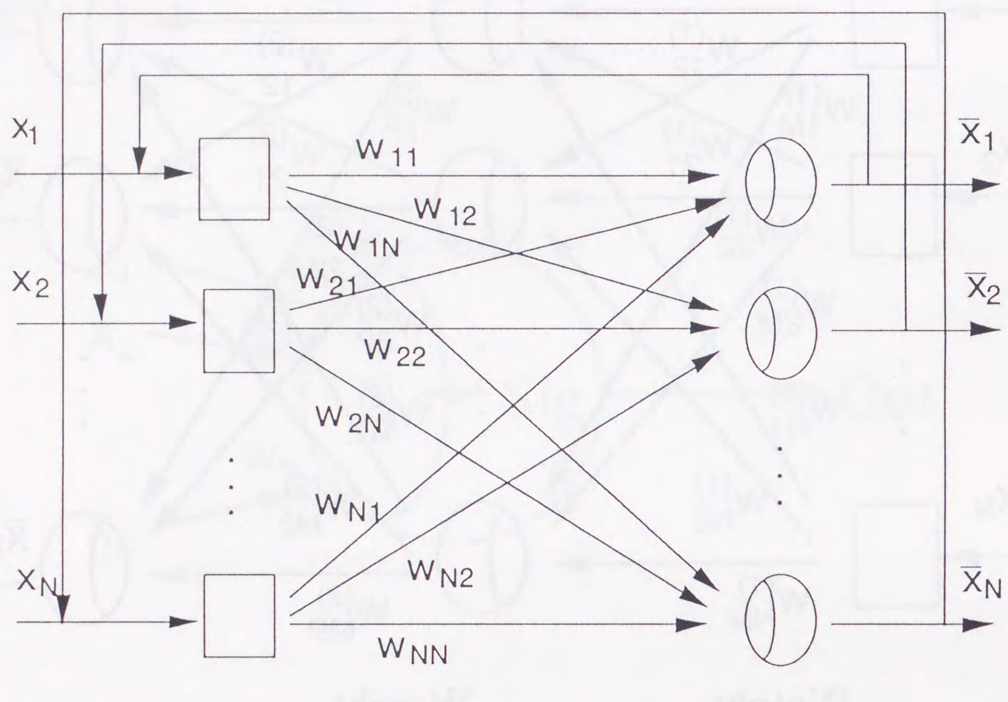



**Figure 2-3:** An associative memory model with feedback

neurons are done at only one layer with feedback, the network is called connected neural network, or called associative memory model with feedback. Figure 2-3 shows an associative memory model with feedback. This research mainly studies discrete associative memory model with feedback, sometimes called associative memory model for simplicity.

For an associative memory model with feedback, the first requirement is that the output series in its recalling processes are convergent for any initial input. In other words, the model must be globally stable.



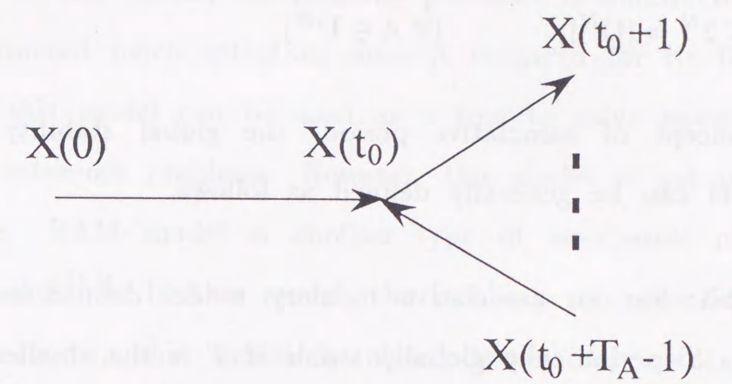



**Figure 2-4:** Cycle limit of series

According to the equation (2.3), the state space of the model given by equations (2.4) and (2.5) is set $U^N$, a finite set. Further, for an arbitrary associative memory model defined by equations (2.4) and (2.5), the outputs in its recalling processes $X(t+1)$ are uniquely determined by inputs $X(t)$ $(t = 0, 1, \cdots)$. Therefore, as shown in Figure 2-4, the output series in its recalling processes must converge to a cycle limit for any initial input, because the number of elements in $U^N$ is finite. That is, for the output series $\{X(t)\}$ on any initial input $X(0) = A$ in the recalling processes of this model, there exists a time $t_0$ and a positive integer $T_A$ such that,

$$X(t + T_A) = X(t) \qquad (\text{for all } t \geq t_0)$$

Clearly, the positive integer $T_A$ is dependent on the initial input $A$, and integers $rT_A$ $(r = 1, 2, \cdots)$ also satisfy the above equation if $T_A$ is the integer such that the above equation is true. Here, for any initial input $A \in U^N$, the smallest positive integer $T_A$ is termed associative period on $A$ in this model. Obviously, for any $A \in \mathbf{U}^N$, the associative period $T_A$ satisfies

$$T_A \leq 2^N = |\mathbf{U}^N|_s \qquad (\forall\, A \in \mathbf{U}^N)$$

With the concept of associative period, the global stability of associative memory model can be generally defined as follows.

**Definition 2-1:** For an associative memory model defined on set $\mathbf{U}^N$, the model is $T$-periodically globally stable if $T$ is the smallest upper bound of the associative period for all patterns in $\mathbf{U}^N$, namely,

$$T = \max_{A \in \mathbf{U}^N} T_A.$$

In particular, we call the model to be globally stable if $T = 1$.

As we have seen, associative memory model with feedback is determined by the connection weight matrix $\mathbf{W}$, function $F_{\mathbf{W}}$ and threshold vector $\vartheta$. Hence, the recalling properties and dynamic behavior of associative memory model depend on the determination of connection weight matrix $\mathbf{W}$, function $F_{\mathbf{W}}$ and threshold vector $\vartheta$. In the conventional associative memory models, such as Hopfield model, function $F_{\mathbf{W}}$, like in equation (1.1), is taken to be a linear combination of the inputs, and the threshold vector is considered as constant, say, $\vartheta_i(t) = 0$ for all $1 \leq i \leq N$ and $t = 0, 1, \cdots$.

## 2.2 Discussion on Conventional AM Models

Up to now, a great number of associative memory models have been proposed. These models themselves have their good properties and defects, respectively. Hopfield model and BAM model can be considered as representatives of associative memory models which have been well studied.



Hopfield model is a type of associative memory model proposed by J.Hopfield[81]. In this model, the recalling processes is unidirectional. Hopfield model has attracted much attention since it occurred for its latent abilities. In particular, this model can be used as a tool to solve pattern recognition and traveling salesman problems. However, this model is not unconditionally globally stable. BAM model is another type of associative memory model given by B.Kosko[111]. In this model, the recalling processes is bidirectional. Like Hopfield model, this model is also powerful for error correction in the pattern recognition. However, unlike Hopfield model, BAM model is unconditionally globally stable because the recalling processes in this model are bidirectional. But, the outputs in the recalling processes of this model are ambiguous because there are two separate neuron fields in it and the recalling can start from any one of the two neuron fields.

Next, we shall discuss these two associative memory models in detail and give concrete examples to explain the problems in these two models.

### 2.2.1 Example of Unstable Hopfield Models

Consider the Hopfield model in which the number of neurons is $N$. Let the learned patterns in this model be $A^{(1)}, \cdots, A^{(m)} \in \mathbf{U}^N$ and assume that they are mutually independent and subject to uniform distribution in state space $\mathbf{U}^N$, where $m$ is the number of learned patterns. Then, based on the Hebbian learning principle, the connection weight between neuron $i$ and $j$ in this model is defined as

$$w_{ij} = \sum_{\xi=1}^{m} A_i^{(\xi)} A_j^{(\xi)}, \qquad (i, j = 1, \cdots, N)$$

In other words, writing it in the matrix form, the connection weight matrix in Hopfield model is

$$\mathbf{W} = \sum_{\xi=1}^{m} \mathbf{A}^{(\xi)T}\mathbf{A}^{(\xi)}; \tag{2.6}$$

where $V^T$ indicates the transpose of vector $V$.

When a pattern $A \in \mathbf{U}^N$ is presented, the recalling processes of this model are performed as follows. Firstly, the pattern

$$\mathbf{X}(0) = \mathbf{A}$$

is given to this model as the initial input. The postsynaptic potential at time $t$ $(t = 0, 1, \cdots)$ is calculated as follows,

$$\mathbf{H}(t+1) = \mathbf{X}(t)\mathbf{W} \tag{2.7}$$

and then the state evolutions of neurons are governed by the following law.

$$\mathbf{X}(t+1) = S(\mathbf{H}(t+1)) \qquad (t = 0, 1, \cdots;) \tag{2.8}$$

where, threshold $\vartheta(t) = 0$, and function $S(\cdot)$ is defined as,

$$S(h_i(t+1)) = \begin{cases} 1 & h_i(t+1) > 0 \\ x_i(t) & h_i(t+1) = 0 \\ -1 & h_i(t+1) < 0 \end{cases} \qquad (i = 1, \cdots, N) \tag{2.9}$$

Here, function $S(\cdot)$ has a little difference from function $Sgn(\cdot)$ shown in equation (2.3) at $h_i(t+1) = 0$.

In probabilistic sense, it is ambiguous that a neuron is in active state or resting when the postsynaptic potential is zero. Therefore, it is reasonable to let the states of the neurons be unchanged when their potentials are zero. In Chapter 4, we shall analyze the dynamic behavior of Hopfield model when the absolute value of the postsynaptic potentials at some neurons are small, and then deduce the optimum dynamic threshold in this model for improving its dynamic behavior and enlarging its memory capacity.

The model given by equations (2.7) and (2.8) is called Hopfield model. It is an associative memory model with feedback. As we mentioned previously, the first requirement is to guarantee that this model is globally stable. In other words, it must guarantee that the output series in the recalling processes of this model must be convergent for any initial input. Unfortunately, a great number of examples show that Hopfield model is not unconditionally globally stable.



In order to guarantee that the model is globally stable, J.Hopfield[81] introduced a Lyapunov function

$$\mathbf{E}(t) = -\frac{1}{2}\mathbf{X}(t)\mathbf{W}\mathbf{X}^T(t)\,; \tag{2.10}$$

to this model and gave the following conditions.

**C1:** The state evolutions of the neurons in the recalling processes are asynchronous.

**C2:** The connection weight matrix in this model is symmetric.

**C3:** The diagonal elements of the connection weight matrix are nonnegative.

By proving that the Lyapunov function is decreasing with time $t$ in the recalling processes of this model under the above three conditions, he demonstrated that Hopfield model is globally stable.

This method is very effective for proving that a dynamic system is globally stable, and has been used by many researchers[61],[111]. But the conditions given by J.Hopfield are very stringent and difficult to explain by neurology. Furthermore, none of these conditions can be easily ignored although they are only sufficient. J.Hopfield found the examples[82] in which the recalling output series converge to cycle limits with period 2 in his simulation experiments when the connection weight matrix is unsymmetrical. In fact, the following concrete examples elucidate that, if any one of the conditions given by J.Hopfield is not true, the remaining conditions will become insufficient. Further, for a Hopfield model with $N$ neurons, the periods of cycle limits is not only 2 but can be arbitrary, even larger than or equal to $2N$ when the connection weight matrix is unsymmetrical.

**Example 2.2.1.1:** Let the learned patterns be

$$\alpha_1 = (\ 1\ 1\ 1\ \ 1\ \ 1)$$

$$\alpha_2 = (\ 1\ 1\ 1\ \ 1\ -1)$$

$$\alpha_3 = (\ 1\ 1\ 1\ -1\ -1)$$

$$\alpha_4 = (\ 1\ 1\ 1\ -1\ \ 1)$$

and the connection weight matrix be defined as follows,

$$\mathbf{W_1} = \alpha_1^T\alpha_2 + \alpha_2^T\alpha_3 + \alpha_3^T\alpha_4 + \alpha_4^T\alpha_1 - 4I$$

$$= \begin{bmatrix} 0 & 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$

where $I$ is the identity matrix.

Obviously, the matrix $\mathbf{W_1}$ is unsymmetrical, namely, the condition **C2** is not true but **C3** is true.

In the recalling processes of this model, even though the conditions **C1** is true, for initial input $\alpha_1$, the series of the state evolutions of neurons is

$$\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4 \to \alpha_1 \to \cdots;$$

This is a periodical series with period 4. Thus, the model with connection weight matrix $\mathbf{W_1}$ is 4−periodically globally stable at least.

In general, if the connection weight matrix in the Hopfield model with $N$ neurons is



$$\mathbf{W_{1'}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

then, the model is $2N-$periodically globally stable.

From this construction example, we have seen the sizes of cycle limits can be arbitrary and even larger than or equal to $2N$ in Hopfield model if the condition **C2** is not true.

**Example 2.2.1.2:** Let the connection weight matrix in Hopfield model be

$$\mathbf{W_2} = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * \\ 0 & * & 0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & 0 \end{bmatrix}$$

Clearly, matrix $\mathbf{W_2}$ satisfies the condition **C2** but not **C3**, and the model with this matrix $\mathbf{W_2}$ is at least 2−periodically globally stable, even though the condition **C1** is true in its recalling processes.

**Example 2.2.1.3:** Let the learned patterns be

$$\alpha_1 = (\ 1\ \ 1\ \ 1\ \ \ 1\ \ \ 1\ )$$

$$\alpha_2 = (\ 1\ \ 1\ \ 1\ -1\ -1\ )$$

and the connection wright matrix be constructed as

$$\mathbf{W_3} = \alpha_1^T\alpha_2 + \alpha_2^T\alpha_1 - I_2$$

$$= \begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

where $I_2^T = diag(2\ 2\ 2\ -2\ -2)$, and clearly the conditions **C2** and **C3** are satisfied for matrix $\mathbf{W}_3$.

In the recalling processes of this model with the matrix $\mathbf{W}_3$, let $\alpha_1$ be the initial input, then the output series is

$$\alpha_1 \to \alpha_2 \to \alpha_1 \to \cdots;$$

if the state evolutions of neurons are synchronous. Namely, this model is 2−periodically globally stable, at least, if the condition **C1** is not true.

From these examples, we see that Hopfield model may be not globally stable for some connection weight matrix $\mathbf{W}$ $(\in \mathbf{R}^{N\times N})$. To guarantee that this model is globally stable, it is necessary to restrict the connection weight matrix to satisfy some conditions. In other words, Hopfield model is not an unconditionally globally stable associative memory model.

By using a Lyapunov function, Gloes-Chace et al[61] discussed the global stability of Hopfield model in 1985. They defined the Lyapunov function

$$E(t) = -\frac{1}{2}\mathbf{X}(t)\mathbf{W}\mathbf{X}^T(t+1)$$

and proved that this model is 2-periodically globally stable if the condition **C2** is true. Furthermore, in 1990, J.Bruck[30] explained that Hopfield model is 4-periodically globally stable if its connection weight matrix is anti-symmetric by graph theory.

It is a powerful method using a Lyapunov function as a tool to prove that a dynamic system is globally stable. J.Hopfield[81] first used this method to associative memory model. But, the condition **C1** means losing the feature of parallel processing, one of the fundamental features of brain information processing. Even though the global stability in Hopfield model is weakened to 2-periodical global stability, by the results given by Gloes-Chace et al[61], the condition **C2** is also remained. The condition **C2** is difficult to explain by neurology and it restricts the selection of the connection weight matrix in Hopfield model. In other words, it restricts optimizing this model.



Further, for any initial input, the limit of output series in the recalling processes can not be determined by this method. S.Aiyer et al[4] utilized subspace geometry theory to analyze the distribution of equilibrium states in Hopfield model and proved that spurious equilibrium states can occur at any corner of the hypercube which is on, or near, the subspace spanned by the learned patterns. Consequently, it can not guarantee that the limit of output series in the recalling processes of Hopfield model must be the desired patterns on their initial input, even non-spurious equilibrium states.

### 2.2.2 Remark on BAM model

B.Kosko[110] discussed the conditions which guarantee that Hopfield model is globally stable and proposed a Bidirectional Associative Memory (BAM) model in 1987. Like in Hopfield model, by means of a Lyapunov function, he proved that BAM model is an unconditionally globally stable associative memory model.

BAM model is compound of two groups of neurons called neuron field $A$ and $B$. Within the same field, the neurons are not connected, they are only indirectly connected to one another through the neurons of other field. The connection weight between neurons $i$ in field $A$ and $j$ in field $B$ is written $w_{ij}$, so is it the connection weight between neurons $j$ in field $B$ and $i$ in field $A$.

Consider a BAM model in which the neuron numbers of neuron fields $A$ and $B$ are $N_1$ and $N_2$, respectively, where $N_1$ and $N_2$ may be equal to each other or not. Let the learned patterns in field $A$ be $A^{(1)}, \cdots, A^{(m)} \in \mathbf{U}^{N_1}$, and the learned patterns in field $B$ be $B^{(1)}, \cdots, B^{(m)} \in \mathbf{U}^{N_2}$, respectively. Then, the connection weight $w_{ij}$ is defined as follows.

$$w_{ij} = \sum_{\xi=1}^{m} A_i^{(\xi)} B_j^{(\xi)}; \qquad (i = 1, \cdots, N_1 \text{ and } j = 1, \cdots, N_2)$$

In other words, the matrix form of the connection weights in BAM model can be written as

$$\mathbf{W} = \sum_{\xi=1}^{m} A^{(\xi)^T} B^{(\xi)}. \tag{2.11}$$

When an initial input $V$ is presented, if $V \in \mathrm{U}^{N_1}$, then, the recalling processes are performed by

$$\begin{aligned} \mathbf{X}(0) &= V \\ \mathbf{H}(t) &= \mathbf{X}(t)\mathbf{W} \\ \mathbf{Y}(t) &= S(\mathbf{H}(t)) \qquad (t = 0, 1, \cdots;) \qquad (2.12) \\ \mathbf{G}(t) &= \mathbf{Y}(t)\mathbf{W}^T \\ \mathbf{X}(t+1) &= S(\mathbf{G}(t)) \end{aligned}$$

if $V \in \mathrm{U}^{N_2}$, then, the recalling processes are performed by

$$\begin{aligned} \mathbf{Y}(0) &= V \\ \mathbf{G}(t) &= \mathbf{Y}(t)\mathbf{W}^T \\ \mathbf{X}(t) &= S(\mathbf{G}(t)) \qquad (t = 0, 1, \cdots;) \qquad (2.13) \\ \mathbf{H}(t) &= \mathbf{X}(t)\mathbf{W} \\ \mathbf{Y}(t+1) &= S(\mathbf{H}(t)) \end{aligned}$$

where, function $S(\cdot)$, defined in equation (2.9), is a component-wise function of vector.



Hence, suppose $\mathbf{X}(0)$ enters BAM model as initial input, how this model equilibrates to the bidirectional equilibrium state $(\mathbf{X}_e, \mathbf{Y}_e)$ can be formally represented as

$$\begin{array}{ccccc} \mathbf{X}(0) & \longrightarrow & \mathbf{W} & \longrightarrow & \mathbf{Y}(0) \\ \mathbf{X}(1) & \longleftarrow & \mathbf{W}^T & \longleftarrow & \mathbf{Y}(0) \\ \mathbf{X}(1) & \longrightarrow & \mathbf{W} & \longrightarrow & \mathbf{Y}(1) \\ \vdots & & & & \\ \mathbf{X}_e & \longrightarrow & \mathbf{W} & \longrightarrow & \mathbf{Y}_e \\ \mathbf{X}_e & \longleftarrow & \mathbf{W}^T & \longleftarrow & \mathbf{Y}_e \\ \vdots & & & & \end{array}$$

Here, the output series $\{\mathbf{X}(t)\}$ in the neuron field $A$ and $\{\mathbf{Y}(t)\}$ in the field $B$ can be rewritten as $\{\mathbf{X}(0), \mathbf{Y}(0), \mathbf{X}(1), \mathbf{Y}(1), \cdots, \mathbf{X}(t), \mathbf{Y}(t), \cdots\}$. This series is same as that of output series in the recalling processes of Hopfield model if the connection weight matrix $\mathbf{W}$ is symmetric. Therefore, BAM model is an extension of Hopfield model, and has the similar dynamic behavior, approximately. But, the ways of recalling processes between BAM model and Hopfield model are different. In Hopfield model, the recalling processes are only unidirectional, in BAM model, however, they are bidirectional. Because of this, BAM model has a completely different global stability.

Like in Hopfield model, B.Kosko introduced a Lyapunov function to BAM model, which is defined as follows

$$E(t) = -\frac{1}{2}\mathbf{X}(t)\mathbf{W}\mathbf{Y}^T(t).$$

By means of the Lyapunov function, he proved that BAM model is unconditionally globally stable.

It is completely different from the global stability of Hopfield model which holds under the condition shown in the above subsection. This powerful result is due to the bidirectional recalling. When the connection weight matrix is symmetric, the unidirectional recalling is equivalent to the bidirectional recalling. Thus, using this result to elucidate the global stability of Hopfield model, the result given by Gloes-Chace et al[61] can be obtained immediately. In fact, the Lyapunov function introduced by B.Kosko is very similar to one given by Gloes-Chace et al[61]. But, as shown in example 2.2.1.1 in subsection 2.2.1, it can not be demonstrated that Hopfield model is unconditionally globally stable, even though unconditionally 2-periodically globally stable.

For any initial input, the output series in the recalling processes of BAM model converge to the bidirectional equilibrium state $(\mathbf{X}_e, \mathbf{Y}_e)$, where $\mathbf{X}_e$ and $\mathbf{Y}_e$ are equilibrium states at neuron fields $A$ and $B$, respectively. One of the patterns in the bidirectional equilibrium state may be the desired pattern of auto-association and the other may be the desired pattern of hetero-association on its initial input, respectively. Consequently, BAM model can be used as an auto-associative memory model as well as a hetero-associative memory model. Further, when the memory capacity of this model is estimated, both its neuron fields can be taken into account. In every neuron field, the dynamic properties of recalling processes is equivalent to those in Hopfield model. That is, the memory capacity of each neuron field in BAM model is approximately equal to that in Hopfield model. Hence, the memory capacity of BAM model is about two times that of Hopfield model.

But, the existence of two neuron fields in BAM model can lead to a problem. For elucidating this problems, let us see the following example.



**Example 2.2.2.1:** Let the learned patterns be

$$
\begin{aligned}
A^{(1)} &= (\ 1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ ) \\
A^{(2)} &= (\ 1\ \ -1\ \ 1\ \ -1\ \ 1\ \ -1\ \ 1\ \ -1\ ) \\
A^{(3)} &= (\ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1\ )
\end{aligned}
$$

and

$$
\begin{aligned}
B^{(1)} &= (\ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1\ ) \\
B^{(2)} &= (\ -1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ ) \\
B^{(3)} &= (\ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ )
\end{aligned}
$$

in the neuron fields $A$ and $B$ of BAM model, respectively. Then the connection weight matrix is:

$$
\begin{aligned}
\mathbf{W} &= {A^{(1)}}^T B^{(1)} + {A^{(2)}}^T B^{(2)} + {A^{(3)}}^T B^{(3)} \\
&= \begin{bmatrix}
1 & 3 & 1 & 1 & -1 & -1 & 1 & 1 \\
3 & 1 & -1 & -1 & 1 & 1 & 3 & 3 \\
-1 & 1 & -1 & -1 & -3 & -3 & -1 & -1 \\
1 & -1 & -3 & -3 & -1 & -1 & 1 & 1 \\
-1 & 1 & 3 & 3 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & 3 & 3 & 1 & 1 \\
-3 & -1 & 1 & 1 & -1 & -1 & -3 & -3 \\
-1 & -3 & -1 & -1 & 1 & 1 & -1 & -1
\end{bmatrix}.
\end{aligned}
$$

Consider the initial input is:

$$V = (\ 1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ )$$

Obviously, the learned pattern $A^{(1)}$ is the desired pattern on $V$ since $V = A^{(1)}$. Namely, the output series in the recalling processes of this model on the initial input $V$ ought to converge to pattern $A^{(1)}$, or bidirectional equilibrium state $(A^{(1)}, B^{(1)})$.

When the recalling process starts from the neuron field $A$, the output series are

$$\mathrm{V} \longrightarrow \mathrm{W} \longrightarrow \mathrm{B}^{(1)}$$

$$\mathbf{A}^{(1)} \longleftarrow \mathbf{W}^T \longleftarrow \mathbf{B}^{(1)}$$

$$\mathbf{A}^{(1)} \longrightarrow \mathbf{W} \longrightarrow \mathbf{B}^{(1)}$$

$$\vdots$$

$$\mathbf{A}^{(1)} \longrightarrow \mathbf{W} \longrightarrow \mathbf{B}^{(1)}$$

$$\mathbf{A}^{(1)} \longleftarrow \mathbf{W}^T \longleftarrow \mathbf{B}^{(1)}$$

$$\vdots$$

That is, the output series equilibrate to pattern $A^{(1)}$, indeed. The recalling process on pattern $V$ is successful in this case. But, when the recalling process starts from the neuron field $B$, the output series are

$$\mathbf{A}^{(2)} \longleftarrow \mathbf{W}^T \longleftarrow \mathbf{V}$$

$$\mathbf{A}^{(2)} \longrightarrow \mathbf{W} \longrightarrow \mathbf{B}^{(2)}$$

$$\mathbf{A}^{(2)} \longleftarrow \mathbf{W}^T \longleftarrow \mathbf{B}^{(2)}$$

$$\vdots$$

$$\mathbf{A}^{(2)} \longrightarrow \mathbf{W} \longrightarrow \mathbf{B}^{(2)}$$

$$\mathbf{A}^{(2)} \longleftarrow \mathbf{W}^T \longleftarrow \mathbf{B}^{(2)}$$

$$\vdots$$

Instead of $(A^{(1)}, B^{(1)})$, in this case, the bidirectional equilibrium states is $(A^{(2)}, B^{(2)})$. Namely, $A^{(1)}$ is not the equilibrium state in this case and then the recalling process is unsuccessful. In fact, it can not be successful because the pattern $A^{(1)}$ is stored only at neuron field $A$, not neuron field $B$.



Consequently, in order that the desired patterns can be correctly retrieved for any initial input, it must be known that the desired pattern is stored at which of the two neuron fields in BAM model before the recalling processes are begun. This is not reasonable because retrieving the desired pattern on any initial input is just the essential task of associative memory model. In principle, this requirement can not be satisfied in BAM model except for a few special cases. For example, the neuron numbers of the two neuron fields are different. In this case, we can determine that the recalling processes ought to start from which of the two neuron fields , by using the information of the dimension of input patterns.

Neurons in biological nervous system are divided into many groups. The neurons within a group are topologically ordered and connected each other, often by proximity[116]. If the system does not develop any trouble, it is impossible to imagine demanding the teaching signals on the desired patterns to be stored at which of neuron groups before the recalling processes have been done. These imply that the structure, in which the neurons are divided into a few groups where the neurons within a group are not directly connected to each other, in a sense does not agree with that of the biological nervous system, such as the structure of BAM model.

## 2.3 Generalized Associative Memory Model

As mentioned previously, the conventional associative memory model, such as Hopfield model in which the recalling processes are unidirectional, is not unconditionally globally stable. Though BAM model is unconditionally globally stable, the problem shown in subsection 2.2.2 remains in it.

Of course, we can avoid this problem in BAM model by the way of storing the same learned patterns at every neuron field. However, this way will greatly decrease the memory capacity of this model to the same one of Hopfield model, and it can not explain the problem that the structure of BAM model does not agree with the structure of biological nervous system. Consequently, it is necessary and expected to look for a generally effective method for solving these problems.

### 2.3.1 Generalized Associative Memory Model on $U^N$

In order to develop a generally effective method for solving the problems still remaining in the conventional associative memory models, let us go back to discuss the original definition of artificial neural network shown in section 2.1. In the equation (2.1), the postsynaptic potential at each neuron is generally determined by a function $F_W$ of the inputs into the neuron, and the function $F_W$ depends on the connection weight matrix. In the conventional associative memory models, such as Hopfield model and BAM model, the form of function $F_W$ is often taken to be the linear combination of inputs. Here, we consider the function $F_W$ to be a nonlinear function. The function is also dependent on the connection weight matrix, and in a sense it is a combination of the inputs. Concretely, for any matrix $W \in R^{N \times n}$, consider the function as follows.

$$\begin{aligned} H(t+1) &= F_W(X(t)) \\ &= Sgn(X(t)W)W^T \qquad (t = 0, 1, \cdots) \end{aligned} \tag{2.14}$$

where $Sgn(\cdot)$ is the sign function and $X(t)$ $(\in U^N)$ is the input pattern into the network at time $t$. $W^T$ is the transpose of connection weight matrix $W$. Sometimes, $W$ and $W^T$ are also called pre-connection-weight matrix and post-connection-weight matrix, respectively[105],[145]. The dimension of the matrix $W$ is $N \times n$. Number $N$ is the dimension of input pattern, i.e., the number of neurons in this model. Number $n$ is a parameter which is independent of the neurons number $N$. The parameter can be arbitrarily determined in compliance with the tasks that this model is set. We refer to the parameter as the characteristic parameter of this model.



For the potential $H(t+1)$ defined by equation (2.14), as in the conventional associative memory models, the state evolutions of neurons are governed by the following law.

$$X(t+1) = S(H(t+1) - \vartheta(t+1)) \qquad (t = 0, 1, \cdots) \tag{2.15}$$

where $S(\cdot)$ is the function defined in equation (2.9) and $\vartheta(t+1)$ is threshold vector, which may well differ from one element to the next. For simplicity, in the following discussion, the threshold vector is always considered to be zero if no stating. We refer to the model defined by equations (2.14) and (2.15) as the generalized associative memory model. Figure 2-5 shows the flow chart of this model.

In equation (2.14), we can divide the term on its right hand side into the forms of a middle state

$$Y(t) = Sgn(X(t)W) \tag{2.16}$$

and the linear combination of the middle state.

$$H(t+1) = Y(t)W^T \qquad (t = 0, 1, \cdots)$$

Thus, this model is formally similar to BAM model. But, we can not consider the middle state $Y(t)$ to be an output from an neuron field. Like Hopfield model, in fact, this model has only one neuron field with $N$

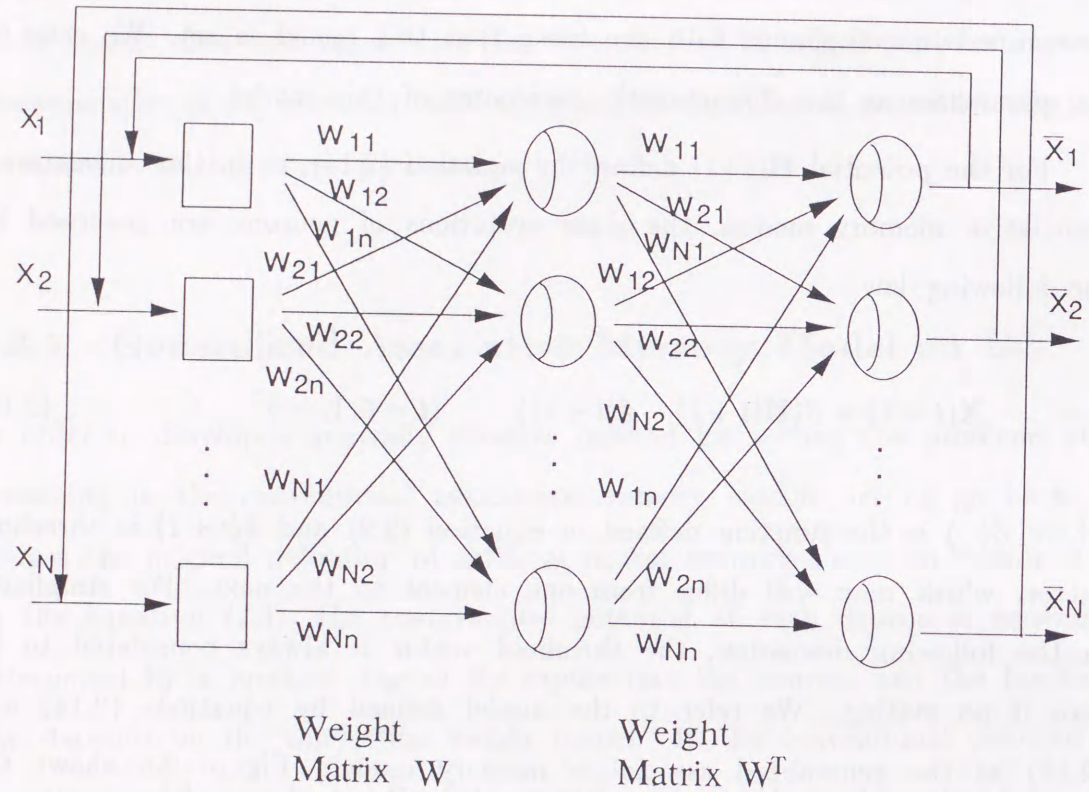



**Figure 2-5:** Flow chart of generalized associative memory model on $U^N$

neurons in which the state are denoted by $\mathbf{X}(t)$. The middle state $\mathbf{Y}(t)$ is completely independent of the learned patterns, and uniquely determined by the connection weight matrix. In other words, the middle state $\mathbf{Y}(t)$ is uniquely determined by the training algorithm for constructing the connection weight matrix. Comparing with the layered neural network model, we can consider that there exists a hidden layer in the generalized associative memory model. The middle state $\mathbf{Y}(t)$ ($\in \mathrm{U}^n$) is the output at time $t$ in the hidden layer. Here, the hidden layer and the set $\mathrm{U}^n$ are called the characteristic layer and characteristic state space of this model, respectively. A dot in



the characteristic layer is called a characteristic site and the number of the characteristic sites is the characteristic parameter of this model. Although we have not made sure whether there exists a middle state in the recalling processes of human nervous system, the existence of the middle state layer can avoid the problem that the neurons within a group are not directly connected to each other.

The recalling processes of this model are done between the neuron field and the characteristic layer as follows and always start and end at the neuron field.

**first:** mapping the inputs into the characteristic state space. In other words, preprocessing the inputs by extracting their characteristics.

**second:** retrieving the desired pattern on the initial input by the characteristics extracted in the first step.

Repeating the processes until the output series equilibrate to the equilibrium state of this model. Hence, in a sense, the recalling processes in this model is bidirectional, but it is not bidirectionally associative.

In the recognition processes of human brains, we always extract some characteristics by observing the given patterns initially, and then judge what the given patterns are by means of these extracted characteristics and our experience. The way to extract the characteristics often directly influences the recognizing effect. In particular, when an unfamiliar pattern is presented, the process of recognition is very obvious. Hence, the bidirectional recalling in this model: extracting characteristics $\rightarrow$ retrieving desired pattern $\rightarrow$ extracting characteristics $\rightarrow$ retrieving desired pattern $\rightarrow \cdots$ can be more naturally explained by the recognition processes of human brains.

We have pointed out that the unconditional global stability of BAM model is due to its recalling processes being bidirectional. The recalling processes of the generalized associative memory model are also bidirectional although the meaning of "bidirectional" is different from one in BAM model. Consequently, the global stability of this model is identical to BAM model.

Like in the conventional associative memory models, we also introduce a Lyapunov function defined as

$$E_a(t) = -\mathbf{X}(t)\mathbf{W}Sgn^T[\mathbf{X}(t)\mathbf{W}] \tag{2.17}$$

to this model and by means of this function, we can obtain the next theorem.

**Theorem 2.3.1:** For any given matrix $\mathbf{W} \in \mathrm{R}^{N \times n}$, the model defined by equations (2.14) and (2.15) is globally stable. That is, for any initial input $V$, the output series $\{\mathbf{X}(t)\}$ in the recalling processes unconditionally converge to the equilibrium state of this model.

**Proof :** To prove this theorem, we can discuss the increments of the Lyapunov function defined in equation (2.17) on time $t$.

$$\begin{aligned}\Delta E_a(t) &= E_a(t+1) - E_a(t) \\ &= -\{\mathbf{X}(t+1)\mathbf{W}Sgn^T(\mathbf{X}(t+1)\mathbf{W}) - \mathbf{X}(t)\mathbf{W}Sgn^T(\mathbf{X}(t)\mathbf{W})\} \\ &= -\{\mathbf{X}(t+1)\mathbf{W}[Sgn(\mathbf{X}(t+1)\mathbf{W}) - Sgn(\mathbf{X}(t)\mathbf{W})]^T \\ &\quad +[\mathbf{X}(t+1) - \mathbf{X}(t)]\mathbf{W}Sgn^T(\mathbf{X}(t)\mathbf{W})\}\end{aligned}$$

by the definition of function $Sgn(\cdot)$, we have

$$\begin{aligned}&\mathbf{X}(t+1)\mathbf{W}[Sgn(\mathbf{X}(t+1)\mathbf{W}) - Sgn(\mathbf{X}(t)\mathbf{W})]^T \\ &= [\mathbf{X}(t+1)\mathbf{W}]\,[\mathbf{Y}(t+1) - \mathbf{Y}(t)]^T \geq 0\end{aligned}$$

and

$$\begin{aligned}&[\mathbf{X}(t+1) - \mathbf{X}(t)]\mathbf{W}Sgn^T(\mathbf{X}(t)\mathbf{W}) \\ &= [\mathbf{X}(t+1) - \mathbf{X}(t)]\mathbf{H}^T(t+1) \geq 0\end{aligned}$$



and this equation is equal to zero if and only if $\mathbf{X}(t) = \mathbf{X}(t+1)$. The sufficiency is obvious, and the necessity can be proved with the definition of function $S(\cdot)$ given in equation (2.9).

Hence, we obtain

$$\Delta E_a(t) \leq 0; \qquad (t = 0, 1, \cdots;)$$

and $\Delta E_a(t) = 0$ if and only if $\mathbf{X}(t) = \mathbf{X}(t+1)$. Namely, in the recalling processes of this model, function $E_a(t)$ is always monotonously decreasing with time $t$ when $\mathbf{X}(t) \neq \mathbf{X}(t+1)$. On the other hand, it is obvious that function $E_a(t)$ is bounded on state space $\mathbf{U}^N$ for an arbitrary connection weight matrix $\mathbf{W} \in \mathrm{R}^{N \times n}$.

Therefore, in the recalling processes, $E_a(t)$ must converge to a local minimum value in a finite time. That is, the output series $\mathbf{X}(t)$ must converge to the points at which $E_a(t)$ has the same value and the value must be a minimum of $E_a(t)$ on $\mathbf{X}(0)$ in state space $\mathbf{U}^N$. For an arbitrary initial input $V$, writing the set of convergent points of the output series in the recalling processes as $D_V$. We prove that it has one element at most.

If it is not true, then there is a pattern $V \in \mathbf{U}^N$, such that $|D_V|_s \geq 2$. Namely, there exists two elements in $D_V$, at least. On the other hand, in accordance with the above discussions, for initial input $V$, we can assume that the output series in the recalling processes reached $D_V$ at time $t = t_0$. Thus, there exist $\mathbf{X}(t_0), \mathbf{X}(t_0+1) \in D_V$ and $\mathbf{X}(t_0) \neq \mathbf{X}(t_0+1)$, such that

$$E_a(t_0) = E_a(t_0+1);$$

Namely,

$$\Delta E_a(t_0) = 0.$$

By the definition of function $Sgn(\cdot)$, this equation is not true, because

$$\{\mathbf{X}(t_0+1)-\mathbf{X}(t_0)\}\mathbf{W}Sgn^T[\mathbf{X}(t_0)\mathbf{W}] > 0,$$

when $\mathbf{X}(t_0) \neq \mathbf{X}(t_0+1)$.

Therefore, there is one and only one element in set $D_V$ for any $V \in \mathbf{U}^N$.

♯

Consequently, the model defined in equations (2.14) and (2.15) is an associative memory model with middle hidden layer. Like BAM model, it is unconditionally globally stable. However, there are no the problems still remaining in the conventional associative memory models.

### 2.3.2 Multi-Dimensional Generalized AM Model

We have discussed the definition of 1-dimensional associative memory model and proposed a generalized associative memory model with feedback. However, in the pattern recognition and information processing, input patterns and output patterns often are multi-dimensional. For example, in the optical information processing, the inputs and outputs often are 2-dimensional. Therefore, for using the generalized associative memory model to multi-dimensional information processing, it is required to extend 1-dimensional model into multi-dimensions, in particular, 2-dimensions.

Next, we shall mainly consider how to extend the generalized associative memory model on the state space $\mathbf{U}^N$ into the 2-dimensional case, i.e. the state space is $\mathbf{U}^{N_1\times N_2}$. In order to do it, we first introduce a few multiplication operators on matrix.

Let matrices $A, B \in \mathbf{R}^{N_1\times N_2}$, $C \in \mathbf{R}^{n_1\times n_2}$ and $D \in \mathbf{R}^{n_1N_1\times n_2N_2}$, where $n_1$, $n_2$, $N_1$ and $N_2$ are positive integers. Then, we have the following definitions.



**Definition 2.2:** The direct product of matrix '$\otimes$' is defined as:

$$\begin{aligned} U &= A \otimes C \\ &= \begin{pmatrix} a_{11}C & \cdots & a_{1N_2}C \\ \cdots & \cdots & \cdots \\ a_{N_11}C & \cdots & a_{N_1N_2}C \end{pmatrix}, \end{aligned} \tag{2.18}$$

where matrix $U \in \mathbf{R}^{n_1N_1\times n_2N_2}$ is called the direct product of matrix $C$ by matrix $A$.

**Definition 2.3:** The dot product of matrix '$\odot$' is defined as:

$$\begin{aligned} X &= A \odot B \\ &= \sum_{\substack{1\le i\le N_1 \\ 1\le j\le N_2}} a_{ij}b_{ij} \end{aligned} \tag{2.19}$$

where $X \in \mathbf{R}$ is called the dot product of matrix $A$ and matrix $B$.

**Definition 2.4:** The left product of matrix '$<\mathcal{L}>$' is defined as:

$$\begin{aligned} Y &= A <\mathcal{L}> D \\ &= \sum_{\substack{1\le i\le N_1 \\ 1\le j\le N_2}} a_{ij}(*)_{ij} \end{aligned} \tag{2.20}$$

where $(*)_{ij}$ $(i = 1,\cdots,N_1; j = 1,\cdots,N_2)$ are $n_1 \times n_2$ order sub-matrices of matrix $D$. The matrix $Y \in \mathbf{R}^{n_1\times n_2}$ is called the left product of matrix $D$ by matrix $A$.

**Definition 2.5:** The right product of matrix '$<\mathcal{R}>$' is defined as:

$$\begin{aligned} Z &= D <\mathcal{R}> C \\ &= \begin{pmatrix} (*)_{11}\odot C & \cdots & (*)_{1N_2}\odot C \\ \cdots & \cdots & \cdots \\ (*)_{N_11}\odot C & \cdots & (*)_{N_1N_2}\odot C \end{pmatrix}, \end{aligned} \tag{2.21}$$

where $(*)_{ij}$ $(i = 1,\cdots,N_1; j = 1,\cdots,N_2)$ defined in definition 2.4 and '$\odot$' defined in definition 2.2. The matrix $Z \in \mathbf{R}^{N_1\times N_2}$ is called the right product of matrix $D$ by matrix $C$.

By using these notations, like in 1-dimensional case, we can give 2-dimensional generalized associative memory model as follows.

Let matrix $\mathbf{W} \in \mathbf{R}^{N_1 n_1 \times N_2 n_2}$ and $A \in \mathbf{U}^{N_1 \times N_2}$ be an arbitrary initial input. Then,

$$\mathbf{X}(0) = A.$$

the postsynaptic potential in 2-dimensional generalized associative memory model can be defined as follows.

$$\begin{aligned} \mathbf{H}(t+1) &= F_{\mathbf{W}}(\mathbf{X}(t)) \\ &= \mathbf{W} < \mathcal{R} > Sgn(\mathbf{X}(t) < \mathcal{L} > \mathbf{W}) \qquad (t = 0, 1, \cdots) \end{aligned} \tag{2.22}$$

where $Sgn(\cdot)$ is the sign function of the components of variable matrix, and $\mathbf{X}(t)$ is the input at time $t$. The number of neurons in this model is $N_1 \times N_2$, the dimension of input. The characteristic parameter of this model is $n_1 \times n_2$. It can be arbitrarily determined in compliance with the tasks that this model is set to.

For the potential $\mathbf{H}(t+1)$, the state evolutions of neurons are governed by the following law.

$$\mathbf{X}(t+1) = S(\mathbf{H}(t+1) - \vartheta(t+1)) \tag{2.23}$$

where $S(\cdot)$ is the function defined in equation (2.9) and $\vartheta(t+1)$ is a 2-dimensional threshold. We refer to the model defined by equations (2.22) and (2.23) as 2-dimensional generalized associative memory model.

The 2-dimensional generalized associative memory model can be implemented by all optical devices as in the conventional associative memory model[48],[110]. Optical neural computing is a new and very interesting branch of present day scientific research. As for the detail of this model optical implementation is expected to be studied widely.

# Chapter 3

# Fundamentals of Statistical Neurodynamics

We have given the generalized associative memory model with feedback. In this model, the recalling processes are bidirectional but the initial inputting direction is only one and there are no the problems still remaining in Hopfield model and BAM model. The problems are shown in section 2.2. Making use of the Lyapunov function defined in equation (2.17), we have proved that this model is unconditionally globally stable.

Unconditional global stability is a powerful result. The power comes from the independence of its stability on the connection weight matrix and their nonlinear generality. However, it is limited because it does not tell us where the equilibria occur in the state space. We only know that the Lyapunov function decreases to the local minimum in the recalling processes, i.e., the recalling output series converge to the equilibrium states of this model, but do not know what the equilibrium states are. In the recalling processes of an associative memory model, for any initial input $V$, it is expected that the desired pattern on the initial input $V$ can be retrieved. For any initial input, the desired pattern on this initial input is the learned pattern which has the least effective Hamming distance from the initial input.

The equilibrium states of associative memory model can be divided into spurious equilibrium states and non-spurious equilibrium states. The spurious equilibrium states are the equilibrium states but not the learned patterns of associative memory model. The non-spurious equilibrium states are the equilibrium states and also are the learned patterns of associative memory model.

S.Aiyer[4] proved that, in Hopfield model, the spurious equilibrium states can occur at any corner of the hypercube which is on, or near, the subspace spanned by the learned patterns. In general, there are the spurious equilibrium states in an associative memory model, in particular, its connection weights are decided by the correlation product type of training algorithm. The distribution of equilibrium states in the state space of an associative memory model, and discrimination whether the output series in the recalling processes of an associative memory model converge to a spurious or non-spurious equilibrium state for any initial input are very significant and unsolved problems.

On the other hand, some of the learned patterns may be not correctly stored at an associative memory model. That is, a learned pattern may not be an equilibrium state in an associative memory model. Figure 3-1 shows the relation between the learned patterns and the equilibrium states. Here, $A$ indicates the spurious equilibrium states set, $B$ indicates the learned patterns set which are not the equilibrium states, and $C$ indicates the non-spurious equilibrium states set. Thus, any pattern in set $B$ can not be the limit of an output series in the recalling processes of associative memory model. Any pattern in set $A$ can be the limit of some recalling output series but is not desired for its initial input, since the patterns in set $A$ are not the learned patterns.

Therefore, in order that the recalling outputs converge to the desired pattern on an arbitrary initial input, both $|B|_s$ and $|A|_s$ ought to be small as possible, where $|B|_s$ and $|A|_s$ indicate the numbers of elements in sets $A$ and $B$,



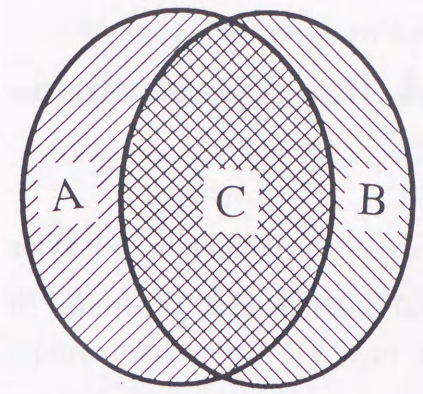



**Figure 3-1.** Relation of the equilibrium states and learned patterns in associative memory model.

respectively. $|B|_s$ can be effectively reduced by restricting the learned pattern number $m$, and improving the training algorithm and the structure of the model. In principle, $|B|_s$ can be reduced to zero if $m$ is sufficiently small relative to the neuron number $N$. However, we do not have an sufficiently effective method to reduce $|A|_s$, although we can get some reduction of $|A|_s$ by improving the training algorithm and the structure of the model. The largest learned pattern number such that the original learned patterns are all correctly stored at an associative memory model is called the memory capacity of this model.

In the recalling processes of an associative memory model, for any initial input $V$, it is expected that the recalling output converge to the desired pattern on the initial input $V$. Next, we shall use statistical neurodynamics to discuss whether the output series in the recalling processes of an associative memory model converge to the desired pattern on their initial input.

## 3.1 Binomial Distribution and its Approximation

In order to use statistical neurodynamics to analyze the dynamic behavior of associative memory model, we first give several fundamental theorems in terms of probability theory and statistics[38],[50].

Repeated independent trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities remain the same throughout the trials. It is usual to denote the two probabilities by $p$ $(0 < p < 1)$ and $1-p$, and to refer to the outcome with probability $p$ as "success", and to other as "failure". Let $b(k;n,p)$ be the probability that a succession of $n$ Bernoulli trials with probabilities $p$ for success and $1-p$ for failure result in $k$ successes and $n-k$ failures. In accordance with the results in probability theory, we have:

$$b(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{3.1}$$

In particular, the probability of nonsuccess is $(1-p)^n$ and the probability of at least one success is $1-(1-p)^n$. Furthermore, for any $0 < k \leq n$, we have

$$\frac{b(k;n,p)}{b(k-1;n,p)} = \frac{(n-k+1)p}{k(1-p)} \tag{3.2}$$

Here, $p$ is considered as a constant. We denote the number of successes in $n$ trials by $S_n$; then $b(k;n,p) = P\{S_n = k\}$. $S_n$ is a random variable and its distribution is $b(k;n,p)$. In the general terminology this distribution is referred to as binomial distribution.

**Remark 3.1** In the Bernoulli trials, the independence among the trials plays a very important role. Generally, it can not be deduced that the distribution of $S_n$ is $b(k;n,p)$ if the condition of independence among the trials is not true.



Usually, we deal with Bernoulli trials where, comparatively speaking, $n$ is large and $p$ is small, whereas the product

$$\lambda = np$$

is of moderate magnitude. In such cases it is convenient to use an approximation to $b(k;n,p)$ which is due to Poisson.

For $b(0;n,p)$, we have

$$\begin{aligned} b(0;n,p) &= (1-p)^n \\ &= \left(1-\frac{\lambda}{n}\right)^n \\ &\approx e^{-\lambda}. \end{aligned}$$

Further, from equation (3.2) it is seen that for any fixed and sufficiently large $n$

$$\frac{b(k;n,p)}{b(k-1;n,p)} = \frac{(n-k+1)p}{k(1-p)} \approx \frac{\lambda}{k},$$

and by induction, we obtain

$$b(k;n,p) \approx \frac{\lambda^k}{k!}e^{-\lambda} \qquad (k = 0, 1, \cdots, n)$$

This is the classical Poisson approximation to the binomial distribution. Introducing the notation $p(k;\lambda)$ defined by

$$p(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda};$$

then $p(k;\lambda)$ indicates a distribution known as Poisson distribution and it is an approximation to $b(k;n,\lambda/n)$ when $n$ is sufficiently large and $\lambda/n$ is small.

We often obtain a good approximation to the binomial distribution by the Poisson distribution when $n$ is large and $p$ is small, whereas $\lambda = np$ is of moderate magnitude. However, when $n \longrightarrow \infty$, a limit distribution of the binomial distribution is necessary. For this, we have the following famous theorem.

**Theorem 3.1:** (*De Moivre-Laplace limit theorem*) For any $z_1, z_2 \in \mathbf{R}$, as $n \longrightarrow \infty$,

$$\mathbf{P}\left\{np + z_1\sqrt{np(1-p)} \leq \mathbf{S}_n \leq np + z_2\sqrt{np(1-p)}\right\} \longrightarrow \Phi(z_2) - \Phi(z_1) \quad (3.3)$$

The limit relation can take on a more pleasing form if $\mathbf{S}_n$ is replaced by the reduced number of successes $\mathbf{S}_n^*$ defined by

$$\mathbf{S}_n^* = \frac{\mathbf{S}_n - np}{\sqrt{np(1-p)}}$$

The inequality on the left side in equation (3.3) is the same as $z_1 \leq \mathbf{S}_n^* \leq z_2$ and hence we can restate equation (3.3) in the form

$$\mathbf{P}\{z_1 \leq \mathbf{S}_n \leq z_2\} \longrightarrow \Phi(z_2) - \Phi(z_1) \quad (3.4)$$

Here, in the general terminology of random variables $np$ is called the expectation, and $np(1-p)$ the variance of $\mathbf{S}_n$, respectively. We also say that $\mathbf{S}_n$ is subject to the normal distribution $N(np, np(1-p))$. The random variable $\mathbf{S}_n^*$ is called standardized form of $\mathbf{S}_n$, and clearly, its expectation and variance are 0 and 1, respectively. Namely, $\mathbf{S}_n^*$ is subject to $N(0,1)$, which is called the standard normal distribution. The function

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$$

is the distribution function of $\mathbf{S}_n^*$, called the standard normal distribution function. The integrand defined by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$



is called the standard normal density function.

As for the normal distribution, we have the following useful lemma which will be used in the next discussion.

**Lemma 3.1** As $x \longrightarrow \infty$

$$1 - \Phi(x) \sim x^{-1}\varphi(x);$$

more precisely, the double inequality

$$\left[x^{-1} - x^{-3}\right]\varphi(x) < 1 - \Phi(x) < x^{-1}\varphi(x)$$

holds for every $x > 0$. Here the sign $\sim$ indicates that the ratio of the two sides tends to one.

## 3.2 A Statistical Variable in Neurodynamics

To discuss whether the output series in the recalling processes of associative memory model approach to the desired pattern on their initial input, we first define a statistical variable, which is termed similar probability in the recalling processes of this model, and then do the discussion by means of the probability. The method using a statistical variable in the neural network to analyze its statistical properties is one of the fundamental methods in the statistical neurodynamics. It has been used into the artificial neural network analyses by many researchers[14],[134],[186], and a number of important results have been obtained.

### 3.2.1 Definition of Similar Probability

For an arbitrary initial input $V$, assuming learned pattern $A^{(\eta)}$ to be the desired pattern on $V$, i.e. the following equation holds,

$$D\left(V, A^{(\eta)}\right) = \min_{1\le i\le m}\left\{D\left(V, A^{(i)}\right)\right\}; \tag{3.5}$$

where $D(\cdot,\cdot)$ is the effective Hamming distance defined on the set $\mathbf{U}^N$. In accordance with the definition of the effective Hamming distance on $\mathbf{U}^N$, we have $D(V, -V) = 0$ for all $V \in \mathbf{U}^N$. Thus, the points $V$ and $-V$ can be considered as one point in a sense, and we can only discuss the case on $V$ if we want to discuss the case on $-V$.

According to the state evolution equation of associative memory model, the output $\mathbf{X}(t) = (x_1(t), \cdots, x_N(t))$ depend on the connection weight matrix, function $F_W(\cdot)$ and threshold for any $\mathbf{X}(0)$ $(\in \mathbf{U}^N)$, where $x_i(t)$ is the output state of neuron $i$ at time $t$. Thus, for arbitrary connection weight matrix, function $F_W(\cdot)$ and threshold, $\mathbf{X}(t) = (x_1(t), \cdots, x_N(t))$ $(t = 0, 1, \cdots)$ is a vector real function defined on the state space $\mathbf{U}^N$. Furthermore, $x_i(t)$ $(1 \le i \le N;\ t = 0, 1, \cdots)$ are random variables because $\mathbf{U}^N$ is a finite set and

$$\left\{V \in \mathbf{U}^N : x_i(t) \le \zeta\right\} \in \mathcal{F} \quad \text{(for each } \zeta \in \mathbf{R})$$

where $\mathcal{F}$ is a $\sigma$–field generated by $\mathbf{U}^N$. A collection $\mathcal{F}$ of subsets of $\mathbf{U}^N$ is called a $\sigma$–field if it satisfies the following conditions[64]:

1. $\emptyset \in \mathcal{F}$;
2. if $V_1, V_2, \cdots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} V_i \in \mathcal{F}$;
3. if $V \in \mathcal{F}$ then $V^c \in \mathcal{F}$, where $V^c$ is the complement of $V$.

Therefore, to discuss whether the output series in the recalling processes of associative memory model approach to the desired pattern on their initial input, we can analyze the probability defined as follows.

$$\begin{aligned} P(t) &= \mathbf{P}\left\{\mathbf{X}(t) = A^{(\eta)}\right\} \\ &= \mathbf{P}\left\{x_1(t) = A_1^{(\eta)}, \cdots, x_N(t) = A_N^{(\eta)}\right\} \qquad (t = 0, 1, \cdots) \end{aligned} \tag{3.6}$$



Clearly, in the probabilistic sense, for any initial input, the output series in the recalling processes converge to the desired pattern on their initial input if and only if the probability $P(t)$ tend to one when $t \longrightarrow \infty$. Further, we have the following lemma.

**Lemma 3.2** In the recalling processes of an associative memory model, if, at time $t_0$, the probability defined in equation (3.6) holds

$$P(t_0 + 1) - P(t_0) > 0; \tag{3.7}$$

then, this model has the error correcting capability or retrieving capability at this time $t_0$, and does not have the capability otherwise.

Consequently, it is sufficient to estimate the probability $P(t)$ and discuss its monotone and its limit as $t \longrightarrow \infty$. However, precise derivation of $P(t)$ is very difficult because the random variables $x_1(t), \cdots, x_N(t)$ are not mutually independent when $t \ne 0$. And so far, the dependence has not been completely elucidated. It is one of the theoretical problems remaining in artificial neural networks, although several research groups try to solve it and have some related useful results[14],[186].

If the dependence between the random variable $x_1(t), \cdots, x_N(t)$ are neglected, then, we can obtain an approximation to $P(t)$ as follows

$$P(t) \approx \prod_{i=1}^{N} \mathbf{P}\left\{x_i(t) = A_i^{(\eta)}\right\} \tag{3.8}$$

Consider the connection weight matrix to be arbitrary in $\mathbf{R}^{N\times N}$, the threshold to be independent of the neuron order and the postsynaptic potential at each neuron to be uniformly calculated with equation (2.4). Then, for arbitrary time $t$, in probabilistic sense, random variable $x_i(t)$ is not dependent on the neuron order and $x_1(t), \cdots, x_N(t)$ have the identical distribution. Thus, we have,

$$\mathbf{P}\left\{x_1(t) = A_1^{(\eta)}\right\} = \cdots = \mathbf{P}\left\{x_N(t) = A_N^{(\eta)}\right\} \triangleq p(t) \tag{3.9}$$

Here we refer to the probability $p(t)$ as similar probability between the output $\mathbf{X}(t)$ and desired pattern $A^{(\eta)}$ on initial input $\mathbf{X}(0) = V$ at time $t$. The similar probability is an important parameter on the dynamic properties of associative memory model with feedback.

Substituting equation (3.9) into equation (3.8), we obtain

$$P(t) \approx p^N(t) \qquad (t = 0, 1, \cdots)$$

Consequently, when the dependence between $x_1(t), \cdots, x_N(t)$ for all $t$ are neglected in an associative memory model, according to the Lemma 3.2, we have the following results.

**Remark 3.2** In the recalling processes of an associative memory model, if

$$p(t_0 + 1) - p(t_0) > 0 \tag{3.10}$$

then, this model has the error correcting capability or retrieving capability at this time $t_0$, and no the capability otherwise. That is, for any initial input, the output series in the recalling processes approach to the desired pattern on the initial input at time $t_0$ if equation (3.10) is true and do not otherwise, in the probabilistic sense.

**Remark 3.3** When $N$ is sufficiently large, the probability $P(t)$ tends to one if and only if the similar probability $p(t)$ tends to one. Furthermore, we have

$$\begin{aligned} P(t) &\approx (p(t))^N \\ &\approx e^{-(1-p(t))N} \end{aligned} \tag{3.11}$$

and when $N \longrightarrow \infty$, the probability $P(t)$ tends to one if and only if

$$(1 - p(t))N \longrightarrow 0 \qquad (t \longrightarrow \infty)$$



The above two results hold in probabilistic sense. According to these two results, we can discuss the monotone and its limit of the similar probability $p(t)$ as $t \longrightarrow \infty$ instead of the discussion of those on the probability $P(t)$. But, like the probability $P(t)$, rigorously estimating the similar probability $p(t)$ is also difficult because of the dependence.

### 3.2.2 Similar probability in Hopfield Model

Now we estimate the similar probability in Hopfield model. To discuss the general statistical properties of this model we consider the learned patterns $A^{(1)}, \cdots, A^{(m)}$ $(\in \mathbf{U}^N)$ to be random and subject to uniform distribution in $\mathbf{U}^N$, i.e.:

$$\mathbf{P}\left\{A_j^{(\xi)} = \pm 1\right\} = \frac{1}{2} \qquad (1 \le j \le N \text{ and } 1 \le \xi \le m). \tag{3.12}$$

Let $\mathbf{X}(0)$ be arbitrary initial input and the desired pattern on $\mathbf{X}(0)$ be $A^{(\eta)}$, a learned patterns. And for simplicity, we only consider the case $\mathbf{X}(0){A^{(\eta)}}^T \ge 0$. When $\mathbf{X}(0){A^{(\eta)}}^T < 0$, the recalling outputs converge to $-A^{(\eta)}$ and the statistical properties are the same properties as in the case $-\mathbf{X}(0){A^{(\eta)}}^T > 0$ in probabilistic sense. Writing

$$(\theta_1(t), \cdots, \theta_N(t)) = \mathbf{X}(t)\mathbf{W} - \mathbf{X}(t){A^{(\eta)}}^T A^{(\eta)} \tag{3.13}$$

Then, like the output state $x_j(t)$, $\theta_j(t)$ $(1 \le j \le N)$, called noise term in the recalling processes, are also random variables which are independent of the subscript $j$ and subject to an identical distribution for all $j$, because the state evolutions of neurons are independent of their location in the network and the connection weight matrix is arbitrary in $\mathbf{R}^{N\times N}$.

By the equation (3.13), the random variable

$$\theta_j(t) = \sum_{1 \le \xi \le m; \xi \ne \eta} \mathbf{X}(t){A^{(\xi)}}^T A_j^{(\xi)}$$

$$= \sum_{1\le\xi\le m;\xi\ne\eta} \sum_{i=1}^{N} x_i(t)A_i^{(\xi)}A_j^{(\xi)} \qquad (1\le j\le N) \tag{3.14}$$

Here, every term in the right hand side of above equation is in either one of the two states: 1 or $-1$. Because the learned patterns satisfy equation (3.12) and the initial input $\mathbf{X}(0)$ is arbitrary, we have

$$\mathrm{P}\left\{x_i(t)A_i^{(\xi)}A_j^{(\xi)} = \pm 1\right\} = \frac{1}{2} \qquad (1\le i,j\le N; 1\le\xi\le m, \xi\ne\eta)$$

Rigorously deriving the distribution of the random variable $\theta_j(t)$ $(1\le j\le N;\ t=1,2,\cdots)$ is difficult because there is the dependence of the recalling outputs on the learned patterns in this model. S.Amari et al[14] tried to analyze this dependence and got some good results. But, up to now, the dependence of the recalling outputs on the learned patterns in an associative memory model has not been clearly elucidated.

If we neglect the dependence of $\mathbf{X}(t)$ on $A^{(\xi)}$ $(1\le\xi\le m, \xi\ne\eta)$, then, for any time $t$, we have $\theta_j(t)$ $(1\le j\le N;\ t=0,1,\cdots)$ are the random variables of $(m-1)N$ Bernoulli trials and the distribution of $\theta_j(t)$ is

$$\mathrm{P}\left\{\theta_j(t) = 2k-(m-1)N\right\}$$

$$\approx b\left(k;(m-1)N,\frac{1}{2}\right) \qquad (0\le k\le (m-1)N)$$

According to the De Moivre-Laplace limit theorem, shown in theorem 3.1, we obtain that, when $N\longrightarrow\infty$,

$$\theta_j^*(t) = \frac{\theta_j(t)}{\sqrt{(m-1)N}} \qquad (1\le j\le N;\ t=0,1,\cdots)$$

is subject to the normal distribution $N(0,1)$, approximately.

Therefore, the similar probability in the recalling processes of Hopfield model is

$$p(t+1) = \mathrm{P}\left\{\mathbf{X}_j(t+1) = A_j^{(\eta)}\right\} \qquad (1\le j\le N)$$



$$= \mathrm{P}\left\{S[\theta_j(t) + \mathbf{X}(t)A^{(\eta)T}A_j^{(\eta)}] = A_j^{(\eta)}\right\}$$

$$= \mathrm{P}\left\{\theta_j(t) < \left|\mathbf{X}(t)A^{(\eta)T}\right|\right\}$$

$$+p(t)\mathrm{P}\left\{\theta_j(t) = \left|\mathbf{X}(t)A^{(\eta)T}\right|\right\}; \tag{3.15}$$

The second term on the right hand side of above equation is equal to zero, and by the definition of standardized effective Hamming distance, we have

$$\left|\mathbf{X}(t)A^{(\eta)T}\right| = \left(1-2D\left(\mathbf{X}(t),A^{(\eta)}\right)\right)N \tag{3.16}$$

Calculating the distance $D(\mathbf{X}(t),A^{(\eta)})$ is difficult for arbitrary initial input. But, we have the expectation

$$E\left\{\left|\mathbf{X}(t)A^{(\eta)T}\right|\right\} = (2p(t)-1)N; \tag{3.17}$$

and, in probabilistic sense, we can utilize the expectation as an estimation of equation (3.16). Substituting the expectation into equation (3.15), the similar probability in Hopfield model is

$$p(t+1) = P\left\{\theta_j(t) < (2p(t)-1)N\right\}$$

$$\approx \Phi\left(\frac{2p(t)-1}{\lambda}\right) \tag{3.18}$$

where the parameter

$$\lambda = \sqrt{(m-1)/N} \tag{3.19}$$

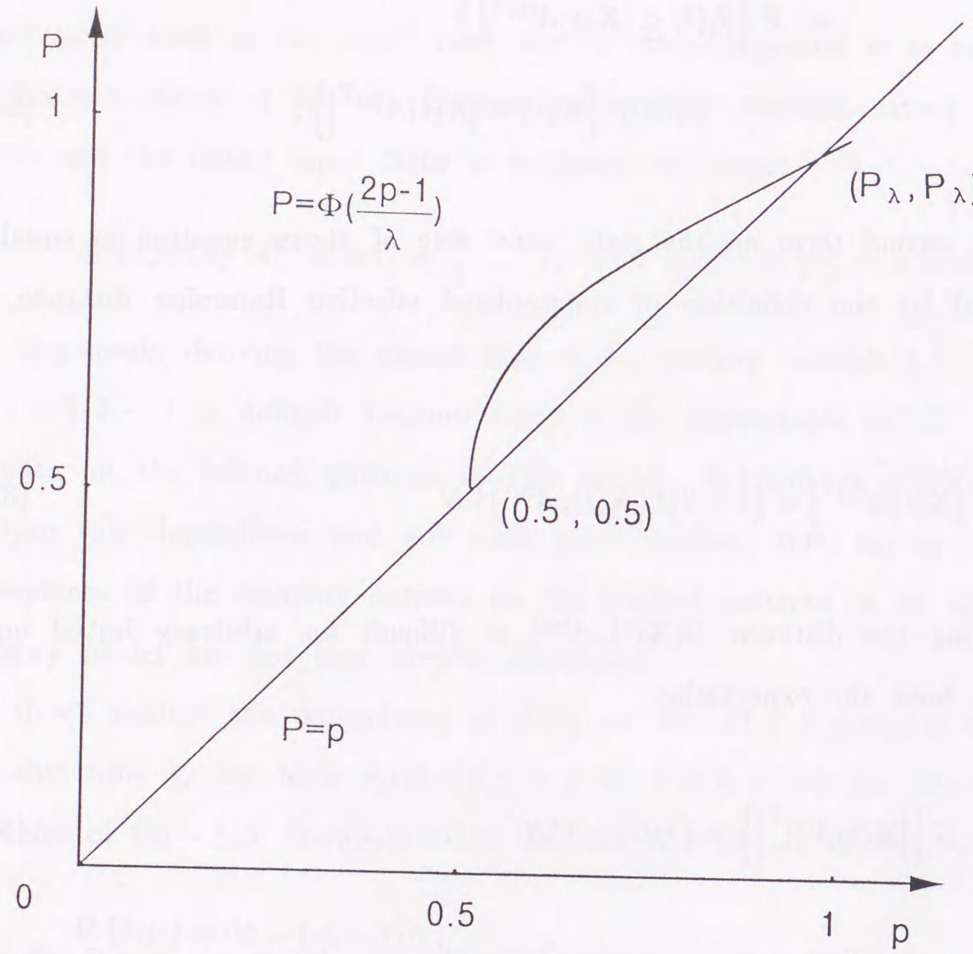



**Figure 3-2.** Image of function $P = \Phi((2p-1)/\lambda)$.

Sometimes, we use the error function $erf(x) = 1 - \Phi(x)$ to denote the similar probability and obtain the similar probability in Hopfield model as

$$p(t+1) \approx 1 - erf\left(\frac{2p(t)-1}{\lambda}\right) \qquad (t = 0, 1, \cdots) \tag{3.20}$$

The curve of the function $P = \Phi[(2p-1)/\lambda]$ is shown in Figure 3-2.



Several researchers[14],[134] deduced this probability in Hopfield model and derived the memory capacity of this model from the probability. But, we must point out that neglecting the dependence between $\mathbf{X}(t)$ and $A^{(\xi)}$ $(1 \le \xi \le m, \xi \ne \eta)$ is not reasonable when the probability $p(t)$ is estimated.

In general, if the dependence of the recalling outputs on the learned patterns are not neglected, the similar probability in an associative memory model is much smaller than that shown in equation (3.20). S.Amari et al[14] and H.Nishimori[151] confirmed this fact in Hopfield model. S.Amari et al[14] discussed the dependence when the probability $p(t)$ is estimated. As usual, they also assumed that $\theta_j(t)$ $(1 \le j \le N)$ are subject to a normal distribution and calculated the expectation and variance of $\theta_j(t)$ $(1 \le j \le N)$. The Result is that, generally, the expectation is not zero and the variance is an increasing function of time $t$, not the constant $(m-1)N$. By taking these into consideration, they obtained a better approximation to the similar probability $p(t)$ in Hopfield model.

However, all of the approximations to the probability $p(t)$ become bad with the increase of learned pattern number[151]. It is one of unsolved theoretical problems in the artificial neural networks how to rigorously estimate the similar probability $p(t)$.

According to the equation (3.20), when the similar probability $p(0)$, called initial similar probability, is given, the similar probability at time $t$ $(> 0)$ in the recalling processes of Hopfield model can be estimated. However, for arbitrary initial input, giving the initial similar probability $p(0)$ is difficult since

$$p(0) = 1 - D\left(\mathbf{X}(0), A^{(\eta)}\right) \tag{3.21}$$

which depends on the initial input. But, we can, in general, only consider the case that the probability $p(0)$ is small, because the similar probability $p(t+1)$ is increasing with $p(t)$ $(> 1/2)$ in the recalling processes of Hopfield model. For example, assume $p(0) = 0.55$.

## 3.3 Capacity of Conventional Associative Memory Model

Using the similar probability, we shall derive some of the statistical properties of conventional associative memory models.

### 3.3.1 A Statistical Upper-Bound on Memory Capacity of Hopfield Model

By means of the Remark 3.2 and similar probability $p(t)$ in Hopfield model, shown in equation (3.20), we can give a statistical upper-bound of the memory capacity of Hopfield model.

In equation (3.20), for any parameter $\lambda\,(>0)$, obviously,

$$p(t+1) \le 1 - erf\left(\frac{1}{\lambda}\right) \qquad (t = 0, 1, \cdots) \tag{3.22}$$

Namely, for any $\lambda\,(>0)$, there exists number $P_\lambda$ $(0 < P_\lambda < 1)$ such that

$$\lim_{t\to\infty} p(t) = P_\lambda < 1 - erf\left(\frac{1}{\lambda}\right)$$

According to Remark 3.3, the number $P_\lambda < 1$ implies that the output series in the recalling processes can not approach to the desired patterns on their initial inputs with probability one in Hopfield model.

Furthermore, by analyzing the function

$$P = 1 - erf\left(\frac{2p-1}{\lambda}\right), \tag{3.23}$$

and its derivative



$$P'_p = \frac{1}{\sqrt{2\pi}} e^{-\frac{(2p-1)^2}{2\lambda^2}} \cdot \frac{2}{\lambda} \tag{3.24}$$

we obtain the next results

- when

$$\sqrt{2/\pi} > \lambda > 0, \tag{3.25}$$

then, there exists a positive number $P_\lambda$ $(> 1/2)$ which is the solution of the equation set

$$\begin{cases} P = 1 - erf[(2p-1)/\lambda] \\ P = p \end{cases} ; \qquad (p > 1/2) \tag{3.26}$$

such that

$$\lim_{t\to\infty} p(t) = P_\lambda\,;$$

and

$$\text{if } p(0) < P_\lambda, \quad \text{then} \quad p(t+1) > p(t),$$
$$\text{if } p(0) > P_\lambda, \quad \text{then} \quad p(t+1) < p(t);$$

- when

$$\lambda \ge \sqrt{2/\pi}, \tag{3.27}$$

equation (3.26) has no solution,

$$\lim_{t\to\infty} p(t) = \frac{1}{2}\,;$$

and the similar probability $p(t)$ is a decreasing function of time $t$. That is, when equation (3.27) is true, the output series $\mathbf{X}(t)$ do not converge to $A^{(\eta)}$ in probabilistic sense.

Therefore, by equation (3.27), we obtain a statistical upper bound of the memory capacity of this model.

$$m < \frac{2}{\pi} N; \tag{3.28}$$

This upper bound is smaller than the upper bound given by Y.Abu et al[1]. They demonstrated that the memory capacity of Hopfield model with $N$ neurons is less than $N$.

**Table 3-1.** Similar probability $p(t)$ $(t = 1, \cdots, 10)$, $P_\lambda$ and $\Phi(1/\lambda)$ in Hopfield model, where $p(0)$ is assumed to be 0.55.

| p(t) | $\lambda$=0.3 | $\lambda$=0.4 | $\lambda$=0.45 | $\lambda$=0.5 | $\lambda$=0.6 | $\lambda$=0.7 |
|---|---|---|---|---|---|---|
| t=1 | 0.631 | 0.599 | 0.587 | 0.579 | 0.562 | 0.556 |
| t=2 | 0.806 | 0.690 | 0.654 | 0.623 | 0.579 | 0.564 |
| t=3 | 0.977 | 0.829 | 0.752 | 0.690 | 0.603 | 0.573 |
| t=4 | 0.999 | 0.950 | 0.868 | 0.779 | 0.633 | 0.582 |
| t=5 | | 0.987 | 0.948 | 0.867 | 0.664 | 0.592 |
| t=6 | | 0.992 | 0.977 | 0.929 | 0.705 | 0.604 |
| t=7 | | | 0.982 | 0.957 | 0.752 | 0.617 |
| t=8 | | | | 0.965 | 0.798 | 0.630 |
| t=9 | | | | 0.967 | 0.840 | 0.644 |
| t=10 | | | | 0.968 | 0.871 | 0.659 |
| $\Phi(\frac{1}{\lambda})$ | 0.99957 | 0.99379 | 0.98671 | 0.97725 | 0.95142 | 0.92210 |
| $P_\lambda$ | 0.99955 | 0.99278 | 0.98310 | 0.97015 | 0.92383 | 0.81530 |



Table 3-1. shows the data of $p(t)$, $\Phi(1/\lambda)$ and $P_\lambda$, where the similar probability $p(0)$ is always assumed to be 0.55. From the table, we can see that the limit value $P_\lambda$ and the convergence rate of similar probability quickly decreases with the parameter $\lambda$ increasing. For example, when $\lambda = 0.3$, $P_\lambda = 0.99955$ and $p(4) = 0.999$, and when $\lambda = 0.7$, $P_\lambda = 0.81530$ and $p(10) = 0.659$.

### 3.3.2 Memory Capacity of Conventional Associative Memory Model

Making use of Lemma 3.1, Remark 3.3 and the approximate expression of similar probability $p(t)$ shown in equation (3.20), the memory capacity of Hopfield model was deduced by several researchers[14],[134]. In fact, by Remark 3.3, the probability that the output series in the recalling processes converge to the desired pattern on their initial input is

$$\begin{aligned} P(1) &\approx \left(1 - erf\left(\frac{1}{\lambda}\right)\right)^N \\ &\approx e^{-N erf(\frac{1}{\lambda})} \end{aligned} \tag{3.29}$$

Using the approximate expression given in Lemma 3.1 into the right hand side of the above equation. the following result was obtained by R.Mceliece et al[134].

**The memory capacity of Hopfield model:** In a Hopfield model with $N$ neurons, if the number of learned patterns is

$$m \le m_c^h = \frac{N}{2 \ln N},$$

then, the learned patterns are all correctly stored, and the output series in the recalling processes converge to the desired pattern on their initial input in probabilistic sense.

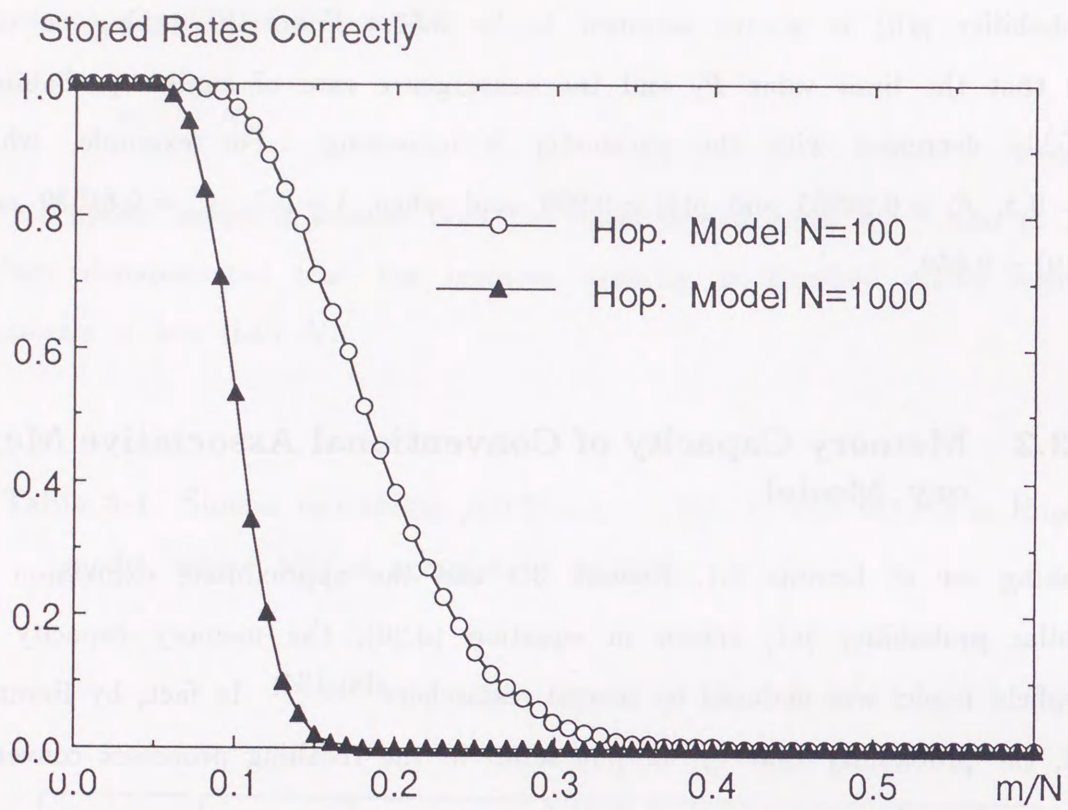



**Figure 3-3.** Rates of the learned patterns storing at Hopfield model.

The parameter $m_c^h$ is called the memory capacity of Hopfield model and $m_c^h/N$ is a decreasing function of the neuron number $N$ and tend to zero as $N \longrightarrow \infty$ in this model. For example, $m_c^h/N \approx 0.11$ when $N = 100$, and $m_c^h/N \approx 0.07$ when $N = 1000$.

Figure 3-3 shows the experimental results of simulations about the rates that the learned patterns are correctly stored at Hopfield model when $N = 100$ and 1000. From the Figure 3-3, we can see that the rates that the learned patterns are correctly stored are about one, when $m/N \leq m_c^h/N$ whenever



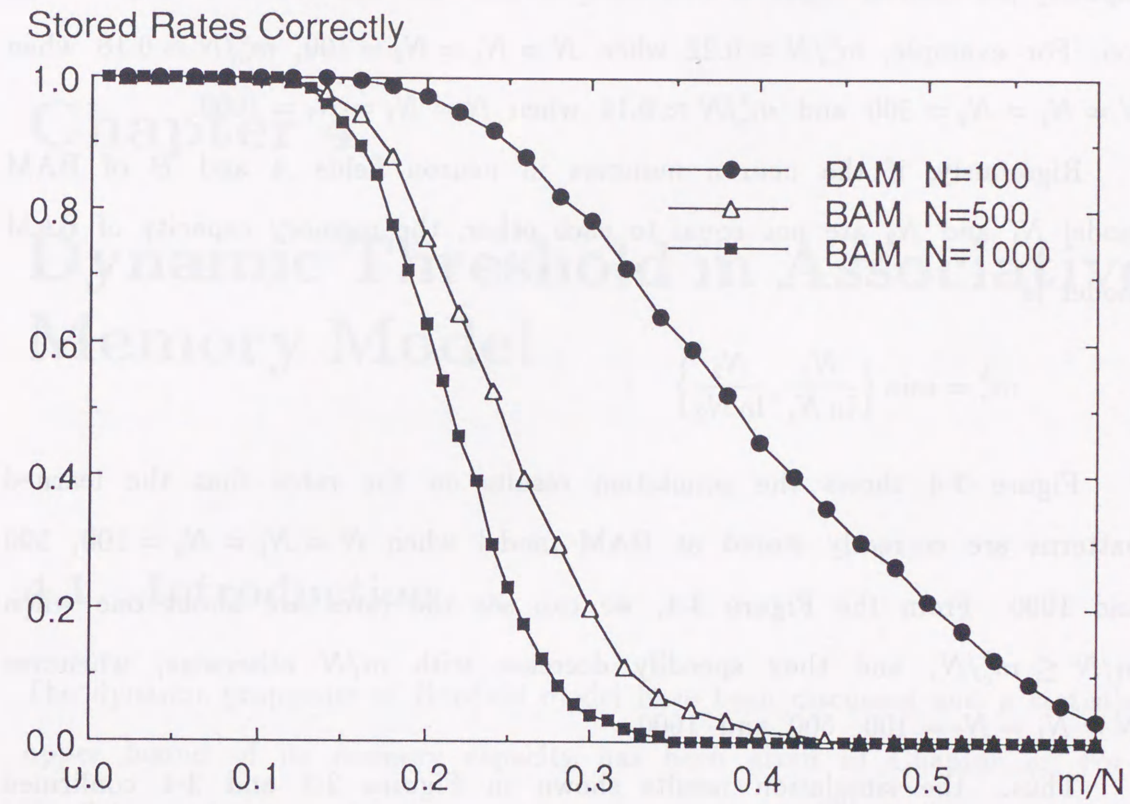



**Figure 3-4.** Rates of the learned patterns storing at BAM model.

$N = 100$ or 1000. Further, when $m/N \geq m_c^h/N$, the rates are all quickly decreasing to zero. Thus. the memory capacities of the Hopfield model is about $m_c^h$. It nearly agrees with the theoretical results.

In the case of BAM model, because there are two neuron fields and every neuron field has the same statistical dynamic properties as in Hopfield model, the memory capacity of BAM model is[111]

$$m_c^b = 2m_c^h = \frac{N}{\ln N}$$

It is also dependent on the neuron number $N$ in this model. The meory capacity per neuron $m_c^b/N$ is decreasing with $N$ and tend to zero as $N \longrightarrow \infty$, too. For example, $m_c^b/N \approx 0.22$ when $N = N_1 = N_2 = 100$, $m_c^b/N \approx 0.18$ when $N = N_1 = N_2 = 500$ and $m_c^b/N \approx 0.14$ when $N = N_1 = N_2 = 1000$.

Rigorously, if the neuron numbers in neuron fields $A$ and $B$ of BAM model $N_1$ and $N_2$ are not equal to each other, the memory capacity of BAM model is

$$m_c^b = \min\left\{\frac{N_1}{\ln N_1}, \frac{N_2}{\ln N_2}\right\}$$

Figure 3-4 shows the simulation results on the rates that the learned patterns are correctly stored at BAM model when $N = N_1 = N_2 = 100$, 500 and 1000. From the Figure 3-4, we can see the rates are about one when $m/N \leq m_c^b/N$, and they speedily decrease with $m/N$ otherwise, whenever $N = N_1 = N_2 = 100$, 500 and 1000.

Thus. the simulation results shown in Figures 3-3 and 3-4 confirmed that the memory capacities of the Hopfield model and BAM model are about $m_c^h$ and $m_c^b$, respectively. This suggests that the theoretical analysis about the memory capacity of conventional associative memory model by using the similar probability is reliable although the dependence is neglected and several approximate expressions is used in the theoretical analysis.

# Chapter 4

# Dynamic Threshold in Associative Memory Model

## 4.1 Introduction

The dynamic properties of Hopfield model have been discussed and a statistical upper bound of its memory capacity has been given in Chapter 3. For a Hopfield model with $N$ neurons, its memory capacity[90] is less than $2N/\pi$. More precisely, its memory capacity[134] is only about $m_c = N/2\ln N$. In other words, the memory capacity per neuron in Hopfield model is

$$\frac{m_c^h}{N} = \frac{1}{2\ln N} \quad \text{(patterns/neuron)}$$

It decreases with $N$ and tend to zero as $N \longrightarrow \infty$. Since the connection weight matrix is square in Hopfield model, the information storage capacity per connection weight is equal to the memory capacity per neuron. Thus, in Hopfield model, $S_c$ is a decreasing function of $N$ and tend to zero when $N \longrightarrow \infty$, too.

Furthermore, the equilibrium state attracting basins in this model are of strange shapes[14], shown in Figure 4-1. That is, the equilibrium state attracting basins hardly contain any spherical neighborhood $O(r)$ $(r > 0)$

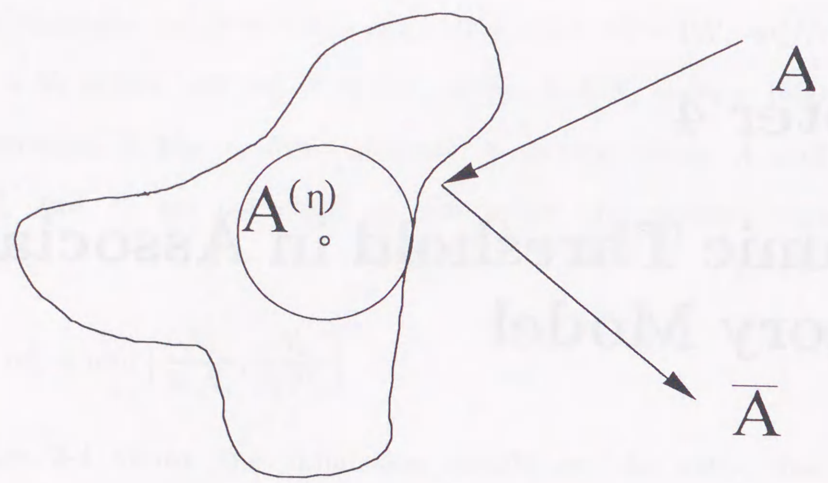



**Figure 4-1:** Strange shape of the attracting basin of equilibrium state in Hopfield model[14].

of the equilibrium state, although the equilibrium state can attract some patterns close to itself in Hopfield model. It is expected to solve these problems and improve the dynamic behavior of Hopfield model.

In this chapter, we shall try to discuss the way how to improve the phenomena of the strange shape attracting basins of equilibrium states, enlarge the size of the non-spurious equilibrium state attracting basins in Hopfield model and increase its memory capacity. Concretely, by means of statistical neurodynamics, we can do as follows;

**d1:** Research methods for increasing the convergent rate of similar probability in the recalling processes of this model and enlarge its limit value as time $t \longrightarrow \infty$, as possible.



**d2:** Research the optimum state evolutions stopping time in the recalling processes so that it can be ended at this time whether the Lyapunov function has reached a minimum or not.

## 4.2 Dynamic Threshold

In Chapter 2, we have explained that the outputs in the recalling processes of Hopfield model are ambiguous when their postsynaptic potentials are equal to zero. This fact is also true when the potentials are approximately equal to zero. This ambiguity always exists in the recalling processes of Hopfield model and it gradually decreases with the output series approaching to the equilibrium state. We try to use this ambiguity by introducing a dynamic threshold to Hopfield model to increase the similarity between the outputs and the desired patterns, i.e., enlarge its similar probability.

### 4.2.1 Definition

Here, we introduce a dynamic threshold[90] to the function $S(\cdot)$ given in equation (2.9) instead of the constant threshold in it. The function with the dynamic threshold is denoted as $\psi(\cdot)$ and defined as follows.

$$\psi[h_i(t)] = \begin{cases} 1 & h_i(t) > c(t) - \alpha(t) \\ x_i(t) & |h_i(t)| \le c(t) - \alpha(t) \\ -1 & h_i(t) < -[c(t) - \alpha(t)] \end{cases} \qquad (i = 1, \cdots, N); \tag{4.1}$$

where $c(t)$ and $\alpha(t)$ are undetermined functions. We refer to the difference $c(t) - \alpha(t)$ as the dynamic threshold in associative memory model. Usually, we consider $c(t) \ge \alpha(t)$.

Clearly, when $c(t) = \alpha(t)$, the function $\psi(\cdot)$ is equivalent to $S(\cdot)$. Therefore, function $S(\cdot)$ is a special instance of function $\psi(\cdot)$. However, function $\psi(\cdot)$ is not a monotonic function. S.Amari[204] et al discussed a type of associative memory model with non-monotonic threshold function and proved that the memory capacity of this model is about $0.39N$. But, unlike function $\psi(\cdot)$, explaining the function given by them is difficult by neurobiology.

We must point out that the method introducing a dynamic threshold holds good in any artificial neural network. In fact, the threshold in biological neural systems is usually variable not constant[11],[66]. It changes with time by a lot of complex objective and subjective factors. Though the fundamental principle of the changes is very involved and has not been completely and clearly explained up to now, it is more reasonable that the threshold in artificial neural network is dynamic than constant. The key is how to determine the dynamic threshold for increasing the efficiency of artificial neural network.

### 4.2.2 Optimum Dynamic Threshold in Hopfield Model

In the recalling processes of Hopfield model, the dynamic threshold $c(t)-\alpha(t)$ is a very important quantity. The dynamic properties of this model are different, when $c(t)-\alpha(t)$ is different. Next, we shall give the optimum dynamic threshold in the sense that increases the efficiency of Hopfield model. The notations defined in the above chapters will be directly applied in the following discussion if no special indication is given.

First, we define

$$c(t) = E\left\{\left| \mathbf{X}(t)A^{(\eta)T} \right|\right\}$$

and by equation (3.17), we have

$$c(t) = (2p(t)-1)N. \tag{4.2}$$



Then, it is only necessary to determine $\alpha(t)$ so that the dynamic threshold $c(t)-\alpha(t)$ is optimum.

Firstly, we shall estimate the similar probability in the recalling processes of Hopfield model with the dynamic threshold $c(t)-\alpha(t)$. For an arbitrary $\alpha(t)$, the similar probability between the output pattern $\mathbf{X}(t+1)$ and the desired pattern $A^{(\eta)}$ on initial input $\mathbf{X}(0)$ is

$$\begin{aligned} p(t+1) &= \mathbf{P}\left\{\mathbf{X}_j(t+1) = A_j^{(\eta)}\right\} \qquad (1 \le j \le N) \\ &= \mathbf{P}\left\{\psi\left[\theta_j(t) + \mathbf{X}(t)A^{(\eta)T}A_j^{(\eta)}\right] = A_j^{(\eta)}\right\} \\ &= \mathbf{P}\{\theta_j(t) < \alpha(t)\} + p(t)P\{\alpha(t) \le \theta_j(t) \le 2c(t)-\alpha(t)\} \end{aligned} \tag{4.3}$$

where random variable $\theta_j(t)$ is defined in equation (3.13), $p(t)$ is the similar probability at time $t$. According to the discussion in Chapter 3, we have random variable $\theta_j(t)$ $(1 \le j \le N)$ are subject to the identical normal distribution $N(0, \lambda^2 N^2)$ and hence, for any $\alpha(t)$, the similar probability is

$$p(t+1) = \hat{\Phi}[\alpha(t)] + p(t)\{\hat{\Phi}[2c(t)-\alpha(t)] - \hat{\Phi}[\alpha(t)]\} \tag{4.4}$$

$$(1/2 < p(t) \le 1, \quad \alpha(t) \le c(t))$$

where $\hat{\Phi}(x)$ is the normal distribution function defined as follows.

$$\hat{\Phi}(x) = \frac{1}{\sqrt{2\pi}\lambda N}\int_{-\infty}^{x} e^{-\frac{t^2}{2\lambda^2 N^2}}\,dt\ ;$$

Unlike in equation (3.15), the second terms in the right hand side of equation (4.4) is not equal to zero when $c(t) \ne \alpha(t)$ and it can not be neglected.

Since $c(t)$ is uniquely determined by the similar probability $p(t)$ in equation (4.2), the similar probability $p(t+1)$ can be considered as a function of $p(t)$ and undetermined variable $\alpha(t)$. Rewriting

$$p(t+1) = P[p(t), \alpha(t)] \qquad (t = 0, 1, \cdots;) \tag{4.5}$$

and then, for an arbitrary $p(t)$, in order that the similar probability $p(t+1) = P[p(t), \alpha(t)]$ is maximum, we can solve an $\alpha_0(t)$ such that the similar probability $P[p(t), \alpha_0(t)]$ is maximum.

In equation (4.4), let $\alpha(t) = c(t)$, we have

$$\begin{aligned} P[p(t), c(t)] &= \hat{\Phi}[c(t)] \\ &= 1 - erf\left(\frac{2p(t)-1}{\lambda}\right) \end{aligned} \tag{4.6}$$

This expression has been obtained in equation (3.20), which is the similar probability in the recalling processes of Hopfield model in which the threshold is zero.

Further, the derivative $P'_\alpha$ of function $P(p, \alpha)$ on $\alpha$ is

$$P'_\alpha(p, \alpha) = \hat{\Phi}'_\alpha(\alpha) - p(t)\left[\hat{\Phi}'_\alpha(\alpha - 2c) + \hat{\Phi}'_\alpha(\alpha)\right]$$

and when $\alpha(t) = c(t)$, we have

$$P'_\alpha[p(t), c(t)] = [1 - 2p(t)]\,\hat{\Phi}'_\alpha[c(t)] < 0\,, \tag{4.7}$$

because

$$\frac{1}{2} < p(t) < 1$$

and

$$\hat{\Phi}'_\alpha[c(t)] = \frac{1}{\sqrt{2\pi}\lambda N} e^{-\frac{c^2(t)}{2\lambda^2 N^2}} > 0\,;$$

Therefore, by equation (4.6) and (4.7), there exists $\alpha(t)\ (< c(t))$ such that

$$P[p(t), \alpha(t)] > P[p(t), c(t)]\,; \tag{4.8}$$



Next, for an arbitrary $p(t)$, we shall evaluate the $\alpha_0(t)$ on which function $P[p(t), \alpha(t)]$ is maximum. Let

$$P'_\alpha[p(t), \alpha(t)] = 0$$

then, we obtain

$$\begin{aligned} \frac{1-p(t)}{p(t)} &= \frac{\hat{\Phi}'_\alpha[\alpha(t) - 2c(t)]}{\hat{\Phi}'_\alpha[\alpha(t)]} \\ &= \exp\left\{-\frac{2c(t)[c(t)-\alpha(t)]}{\lambda^2 N^2}\right\}\,; \end{aligned} \tag{4.9}$$

Clearly, the equation (4.9) has one and only one solution. Here, we use the first degree Taylor expansion of equation (4.9)

$$\frac{1-p(t)}{p(t)} = 1 - \frac{2c(t)[c(t)-\alpha(t)]}{\lambda^2 N^2}$$

to solve an approximate solution of the optimum function $\alpha(t)$ and obtain the solution

$$\alpha_0(t) = c(t) - \frac{\lambda^2 N}{2p(t)}. \tag{4.10}$$

Consequently, in equation (4.1), we can define $\alpha(t)$ as $\alpha_0(t)$, and, in probabilistic sense, the dynamic threshold

$$c(t) - \alpha_0(t) = \frac{\lambda^2 N}{2p(t)}$$

is optimum in Hopfield model. It is a decreasing function of the similar probability. Thus, when Hopfield model has the error correcting capability, i.e. the similar probability $p(t)$ is increasing, the optimum dynamic threshold is decreasing with the probability $p(t)$. This is consistent with the explanation of neurobiology.

In the following discussion, for simplicity, we refer to the Hopfield model with the optimum dynamic threshold $c(t) - \alpha_0(t)$ as optimum dynamically associative memory (ODAM) model.

## 4.3 Dynamic Behavior of ODAM Model

The threshold plays an important role in the neural network. A neural network will have different dynamic behavior when the threshold is different. From the above discussion we have seen that the increasing rate of the similar probability is enlarged in ODAM model. That is, for arbitrary $p(t)$ and $\lambda$, the similar probability $p(t+1)$ in the recalling processes of ODAM model is larger than that in Hopfield model. Next, we shall prove that, for arbitrary parameter $\lambda$, the limit value of the similar probability $p(t+1)$ in ODAM model is also larger than it in Hopfield model, but, it does not tend to one $(t \to \infty)$, either.

### 4.3.1 Converging Properties of ODAM Model

For convenient, we rewrite

$$g(p(t)\,;\lambda) = P\,[\,p(t), \alpha_0(t)\,]; \tag{4.11}$$

then, for any parameter $\lambda\,(>0)$, it is not difficult to prove that function

$$g(p\,;\lambda) = (1-p)\Phi\left(\frac{2p-1}{\lambda} - \frac{\lambda}{2p}\right) + p\Phi\left(\frac{2p-1}{\lambda} + \frac{\lambda}{2p}\right) \tag{4.12}$$

is a monotone increasing function of the variable $p$ and

$$g(p(t)\,;\lambda) \le 1 - erf\left(\frac{1}{\lambda} + \frac{\lambda}{2}\right) \qquad (t = 1, 2, \cdots) \tag{4.13}$$

where $\Phi$ is the standardized normal distribution function. The image of function $g(p\,;\lambda)$ is shown in Figure 4-2.



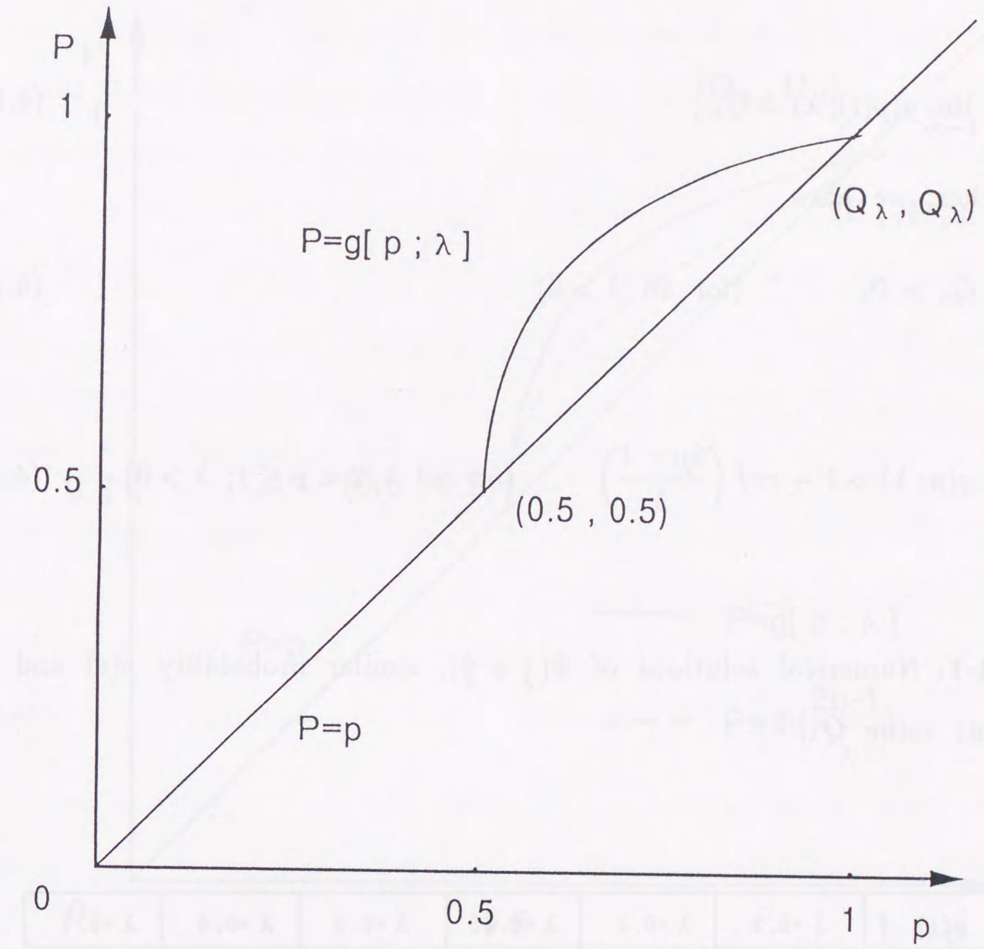



**Figure 4-2:** Image of function $g(p, \lambda)$.

By Figure 4-2, it is obvious that there exists a positive number $Q_\lambda\,(<1)$ which is the solution of the equation set

$$\left\{\begin{array}{l} P = g(p\,;\lambda) \\ P = p \end{array}\right. ; \tag{4.14}$$

such that

$$\text{if} \quad p(0) < Q_\lambda, \qquad \text{then} \qquad p(t+1) = g(p(t);\lambda) > p(t),$$

$$\text{if} \quad p(0) > Q_\lambda, \qquad \text{then} \qquad p(t+1) = g(p(t);\lambda) < p(t),$$

and

$$\lim_{t\to\infty} g(p(t);\lambda) = Q_\lambda\,; \tag{4.15}$$

Further, we have

$$Q_\lambda > P_\lambda \qquad (\text{for all } \lambda > 0) \tag{4.16}$$

because

$$g(p\,;\lambda) > 1 - erf\left(\frac{2p-1}{\lambda}\right) \qquad (\text{for all } 1/2 < p \le 1;\ \lambda > 0) \tag{4.17}$$

**Table 4-1:** Numerical solutions of $\Phi(\frac{1}{\lambda}+\frac{\lambda}{2})$, similar probability $p(t)$ and its limit value $Q_\lambda$.

| p(t) | λ=0.3 | λ=0.4 | λ=0.45 | λ=0.5 | λ=0.6 | λ=0.7 |
|---|---|---|---|---|---|---|
| t=1 | 0.637 | 0.606 | 0.597 | 0.590 | 0.577 | 0.569 |
| t=2 | 0.822 | 0.717 | 0.682 | 0.653 | 0.618 | 0.597 |
| t=3 | 0.987 | 0.878 | 0.809 | 0.753 | 0.680 | 0.634 |
| t=4 | 0.999 | 0.981 | 0.935 | 0.871 | 0.760 | 0.687 |
| t=5 | | 0.995 | 0.984 | 0.951 | 0.845 | 0.743 |
| t=6 | | | 0.990 | 0.978 | 0.911 | 0.802 |
| t=7 | | | | 0.984 | 0.946 | 0.856 |
| t=8 | | | | | 0.958 | 0.895 |
| t=9 | | | | | 0.963 | 0.918 |
| t=10 | | | | | | 0.930 |
| $\Phi(\frac{1}{\lambda}+\frac{\lambda}{2})$ | 0.99975 | 0.99653 | 0.99280 | 0.98769 | 0.97530 | 0.96243 |
| $Q_\lambda$ | 0.99972 | 0.99551 | 0.99131 | 0.98540 | 0.96988 | 0.95342 |



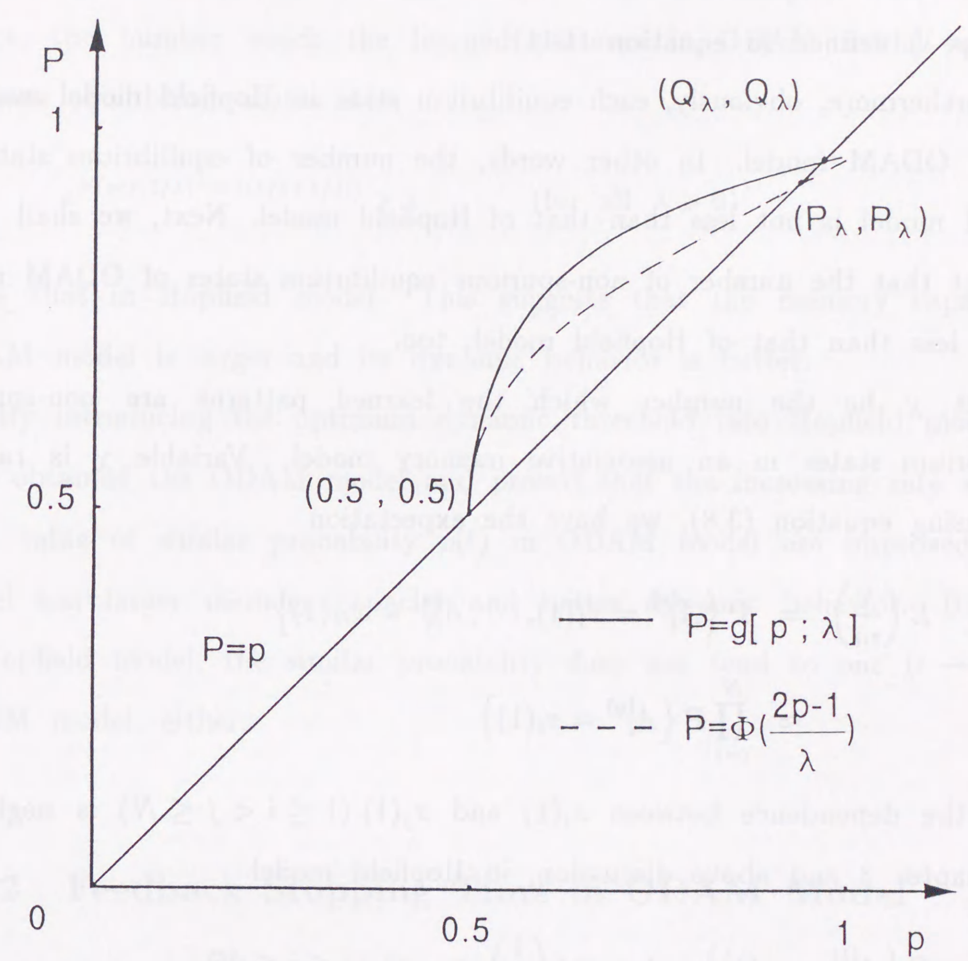



**Figure 4-3:** Images of functions $P = \Phi(\frac{2p-1}{\lambda})$ and $P = g(p,\lambda)$ shown in equation (4.11).

Table 4-1 shows the data of $p(t)$, $\Phi(\frac{1}{\lambda}+\frac{\lambda}{2})$ and $Q_\lambda$. Here, the initial similar probability $p(0)$ is always assumed to be 0.55. By comparing it with Table 3.1, we can see that, when $\lambda$ is large, the increasing rate and the limit value $Q_\lambda$ of the similar probability in ODAM model are much larger than them in Hopfield model. For example, when $\lambda = 0.7$, in Hopfield model,

$p(10) = 0.659$ and $P_\lambda = 0.81530$ but in ODAM model, $p(10) = 0.930$ and $Q_\lambda = 0.95342$. Figure 4-3 shows the images of functions $P = \Phi(\frac{2p-1}{\lambda})$ and $P = g(p;\lambda)$ defined in equation (4.11).

Furthermore, obviously, each equilibrium state in Hopfield model must be one in ODAM model. In other words, the number of equilibrium states of ODAM model is not less than that of Hopfield model. Next, we shall prove the fact that the number of non-spurious equilibrium states of ODAM model is not less than that of Hopfield model, too.

Let $\chi$ be the number which the learned patterns are non-spurious equilibrium states in an associative memory model. Variable $\chi$ is random and, using equation (3.8), we have the expectation

$$E\left(\frac{\chi}{m}\right) = \mathrm{P}\left\{A_1^{(\eta)} = x_1(1), \cdots, A_N^{(\eta)} = x_N(1)\right\}$$

$$\approx \prod_{i=1}^{N} \mathrm{P}\left\{A_i^{(\eta)} = x_i(1)\right\} \qquad (4.18)$$

when the dependence between $x_i(1)$ and $x_j(1)$ $(1 \le i < j \le N)$ is neglected, by Chapter 3 and above discussion, in Hopfield model

$$\mathrm{P}\left\{A_i^{(1)} = x_i(1)\right\} \approx 1 - erf\left(\frac{1}{\lambda}\right) \qquad (1 \le i \le N)$$

and in ODAM model

$$\mathrm{P}\left\{A_i^{(1)} = x_i(1)\right\} \approx 1 - erf\left(\frac{1}{\lambda} + \frac{\lambda}{2}\right) \qquad (1 \le i \le N)$$

Further, $erf(1/\lambda)$ and $erf(1/\lambda + \lambda/2)$ are both very small when $\lambda < \sqrt{2/\pi}$ and equation (4.16) is always true for all $\lambda > 0$. Thus, we obtain the next approximate expressions.

In Hopfield model,

$$E\left(\frac{\chi}{m}\right) \approx e^{-N(erf(1/\lambda))} \qquad (\text{for all } \lambda > 0).$$

And in ODAM model



$$E\left(\frac{\chi}{m}\right) \approx e^{-N(erf(1/\lambda+\lambda/2))} \qquad (\text{for all } \lambda > 0).$$

Hence, the number which the learned patterns in ODAM model are non-spurious equilibrium states is

$$e^{N(erf(1/\lambda)-erf(1/\lambda+\lambda/2))} > 1 \qquad (\text{for all } \lambda > 0) \qquad (4.19)$$

times that in Hopfield model. This suggests that the memory capacity of ODAM model is larger and its dynamic behavior is better.

By introducing the optimum dynamic threshold into Hopfield model, we have obtained the ODAM model and proved that the increasing rate and the limit value of similar probability $p(t)$ in ODAM model are improved. This model has larger memory capacity and better dynamic behavior. But, like in Hopfield model, the similar probability does not tend to one $(t \to \infty)$ in ODAM model, either.

### 4.3.2 Feedback Stopping Time in ODAM Model

The connection weight matrix in ODAM model is also based on Hebbian learning law. That is, it is constructed by the sum of the auto-correlative products of learned patterns. Though it was proved that the increasing rate and the limit value $Q_\lambda$ of similar probability $p(t)$ in ODAM model are larger than those in Hopfield model, we can not guarantee that $p(t)$ tend to one as $t \to \infty$.

Therefore, it is necessary and effective ending the state evolutions in the recalling processes of ODAM model as soon as the similar probability $p(t)$ approximately takes its limit value, so that the state evolutions do not fall down to the spurious equilibrium states. According to the discussion in subsection 4.3.1, for arbitrary parameter $\lambda$, function $g(p;\lambda)$ is a monotone increasing function of $p$, and there exists a positive number $Q_\lambda (> P_\lambda)$ such that

$$\text{if} \quad p(0) < Q_\lambda, \quad \text{then} \quad p(t+1) > p(t),$$

$$\text{if} \quad p(0) > Q_\lambda, \quad \text{then} \quad p(t+1) < p(t);$$

and

$$\lim_{t\to\infty} p(t+1) = \lim_{t\to\infty} g(p(t);\lambda) = Q_\lambda\,; \tag{4.20}$$

Hence, in the recalling processes of ODAM model, for an arbitrary error constant $\delta\ (>0)$, when $p(t+1)$ and $p(t)$ satisfy

$$|\,p(t+1) - p(t)| < \delta\,, \tag{4.21}$$

we can consider that the similar probability has approximately taken its limit value at this time. In other words, the recalling processes of this model can be ended at this time whether they have reached the equilibrium states or not. We refer to this time as the optimum feedback stopping time in the recalling processes of ODAM model.

But, it is difficult usually and not worth to evaluate the value of function $p(t+1) = g(p(t);\lambda)$ by equation (4.11), we can utilize a linear function instead of function $g(p(t);\lambda)$ to estimate an approximate value of $p(t+1)$. For example, the straight line through the points $(Q_\lambda, Q_\lambda)$ and $(1, g(1;\lambda))$

$$P - Q_\lambda = \frac{g(1;\lambda) - Q_\lambda}{1 - Q_\lambda}[\,p(t) - Q_\lambda\,], \tag{4.22}$$

where $g(1;\lambda)$ is

$$g(1;\lambda) = 1 - erf\left(\frac{1}{\lambda} + \frac{\lambda}{2}\right)\,;$$

In equation (4.22), let $P = p(t+1)$, then, we have the approximate expression of the similar probability



$$p(t+1) \approx Q_\lambda + \left[1 - erf\left(\frac{1}{\lambda} + \frac{\lambda}{2}\right) - Q_\lambda\right]\frac{p(t) - Q_\lambda}{1 - Q_\lambda} \tag{4.23}$$

where $Q_\lambda$ can be evaluated by equation (4.14), when the ODAM model is established. Thus, in the recalling processes, we can simply evaluate dynamic threshold $c(t) - \alpha_0(t)$ by the approximate value of $p(t)$, and end the state evolutions when equation (4.21) is satisfactory in the recalling processes for given error constant $\delta$.

However, there is a trouble in the real recalling processes as well. We must know $p(0)$ which is the initial similar probability given by equation (3.21), because dynamic threshold at time $t = 0$ must be evaluated for any initial inputs. We can not accurately know it for arbitrary initial input. But, in probabilistic sense, we can assume that $p(0)$ is a small constant $(1 \geq p(0) \geq 0.5)$. For example, assuming $p(0) = 0.55$.

When the dynamic behavior of an associative memory model is discussed, it is sufficient only considering the case in which the initial input has relatively great effective Hamming distance from its desired pattern, i.e., $p(0)$ is small. By the discussion in subsection 4.3.1, if $p(0)\,(> Q_\lambda)$ is large, we have $p(t) > Q_\lambda\ (t = 0, 1, \cdots;)$ and $p(t) \to Q_\lambda (t \to \infty)$. Thus, the output series in the recalling processes will approach to the desired pattern with a large probability. Even some output series do not converge to the desired patterns, in probabilistic sense, they can not converge to ones having large effective Hamming distance from the desired patterns, as the state evolutions in ODAM model are ended at the optimum feedback stopping time. Therefore, there is no trouble that we only consider the case in which $p(0)$ is small.

### 4.3.3 Simulation Experiments

By introducing the optimum dynamic threshold into Hopfield model, we have proposed an improved Hopfield model which is called ODAM model. The increasing rate and limit value $Q_\lambda$ of similar probability in the recalling processes of ODAM model are larger. Namely, the dynamic behavior of ODAM model is better than that of Hopfield model in probabilistic sense.

Furthermore, we discussed the optimum feedback stopping time in the recalling processes of ODAM model in subsection 4.3.2 and gave the method to estimate the optimum feedback stopping time. Therefore, in ODAM model, the state evolutions are ended at this optimum time not the time when a Lyapunov function has taken local minimum. Thus, the phenomena that the attracting basins of equilibrium states are of strange shapes should be greatly reduced in the recalling processes of ODAM model.

In order to examine these theoretical analysis results and compare the dynamic behavior of ODAM model with those of the Hopfield model, the simulation experiments on these model are done. Because solving equation set (4.14) to obtain $Q_\lambda$ is relatively complex for an arbitrary parameter $\lambda$, in our simulations, we directly utilized a straight line through the points $(0.6, P_1)$ and $(1, P_2)$

$$p(t+1) = P_1 + 2.5(P_2 - P_1)\,[p(t) - 0.6] \tag{4.24}$$

instead of the straight line shown in equation (4.23) for estimating the similar probability, where

$$P_1 = \Phi\left(\frac{1}{\lambda} - \frac{\lambda}{2}\right), \qquad P_2 = \Phi\left(\frac{1}{\lambda} + \frac{\lambda}{2}\right);$$

and they can be obtained from the probability integer table. Of course, it can be done researching an optimum approximate function of $g(p;\lambda)$ instead of the straight line shown in equation (4.24).

We examine the mean of $q^{(i)}(t)$ which is defined as follows.

$$q^{(i)}(t) = 1 - D_t^i \qquad (t = 0, 1, \cdots;\ i = 1, \cdots, \mathcal{N}.) \tag{4.25}$$

where $D_t^i$ is the standard effective Hamming distance between the input



pattern at time $t$ in the recalling processes and the desired pattern on the initial input of the $i$th $(1 \le i \le \mathcal{N})$ test. $\mathcal{N}$ is the number of tests. By equations (3.16) and (3.17), we have

$$E\left[q^{(i)}(t)\right] = p(t) \qquad (t = 0, 1, \cdots) \tag{4.26}$$

Thus, we examine the statistical value of similar probability in the recalling processes of ODAM model. The statistical results of the simulation experiments are shown in Figure 4-4. Here, for examining the similar probability whether depends on the initial inputs, we divide the initial inputs into 11 groups for all tests as follows and show the means of $q^{(i)}(t)$ of every group in Figure 4-4.

Let

$$q_a = 0.5 + 0.05a \qquad (a = 0, 1, \cdots, 10)$$

and using $q^{(i)}(0)$ and $q_a$, we can define the sets

$$I_a = \left\{ i : \left| q^{(i)}(0) - q_a \right| \le 0.02 \right\} \qquad (a = 0, 1, \cdots, 10;\ i = 1, 2, \cdots, \mathcal{N})$$

and divide all the initial inputs in our experiment into 11 groups by the sets $I_a$. Writing the number of elements in $I_a$ as

$$N_a = |I_a|_s$$

Then, for every set $I_a$, the mean of $q^{(i)}(t)$ $(i \in I_a)$ is

$$q_a(t) = \begin{cases} q_a & t = 0 \\ \frac{1}{N_a} \sum_{i \in I_a} q^{(i)}(t) & t \ne 0 \end{cases} \tag{4.27}$$

The neuron number of the ODAM model used in our experiments is $N = 100$. The simulations are done for $m = 20$, 50 and 100, and the results are displayed in Figure 4-4. a, c and e, respectively. We also simulate Hopfield model for $N = 100$, $m = 20$, 50 and 100, and the results are displayed in Figure 4-4. b, d and f, respectively.

These figures suggested that the dynamic properties in the recalling processes of ODAM model agree with the results of theoretical analysis. Even $\lambda$ is so large as one, the output series always approach to the desired patterns on their initial inputs in probabilistic sense. Though the phenomena that the attracting basins of equilibrium states are of strange shapes accidentally appeared in our experiments, the output series do not fall down to the spurious equilibrium states which have a great effective Hamming distance from the desired patterns on their initial inputs.

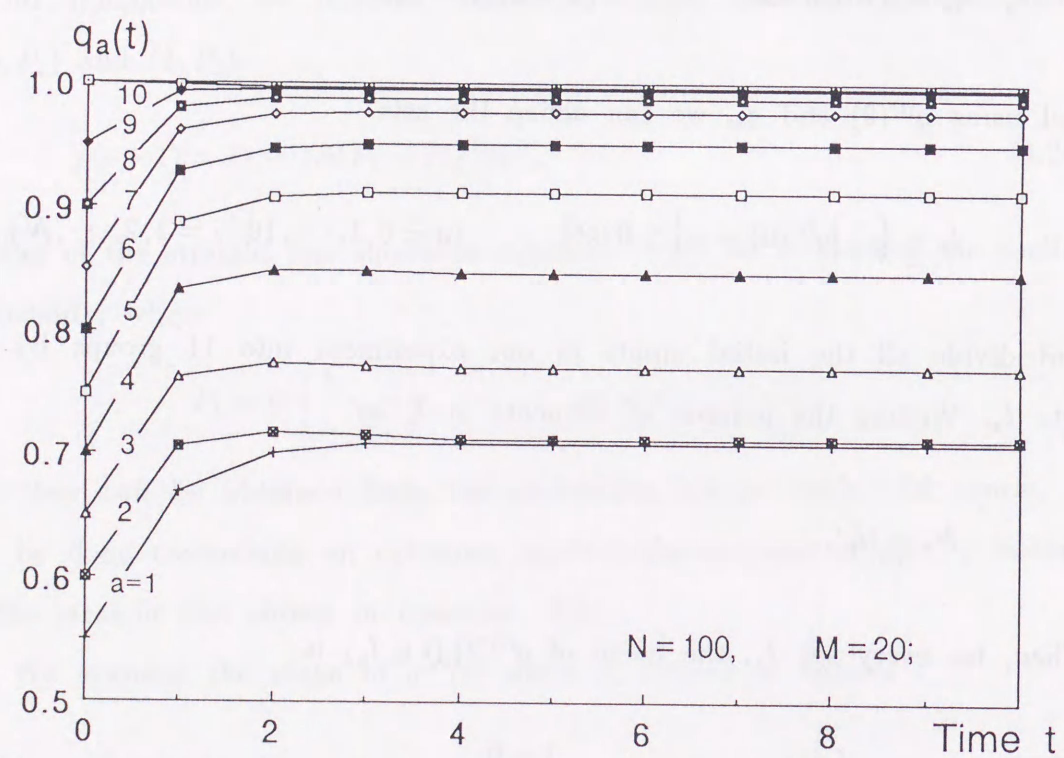



**Figure 4-4. a:** Statistical value on similar probability in ODAM model.



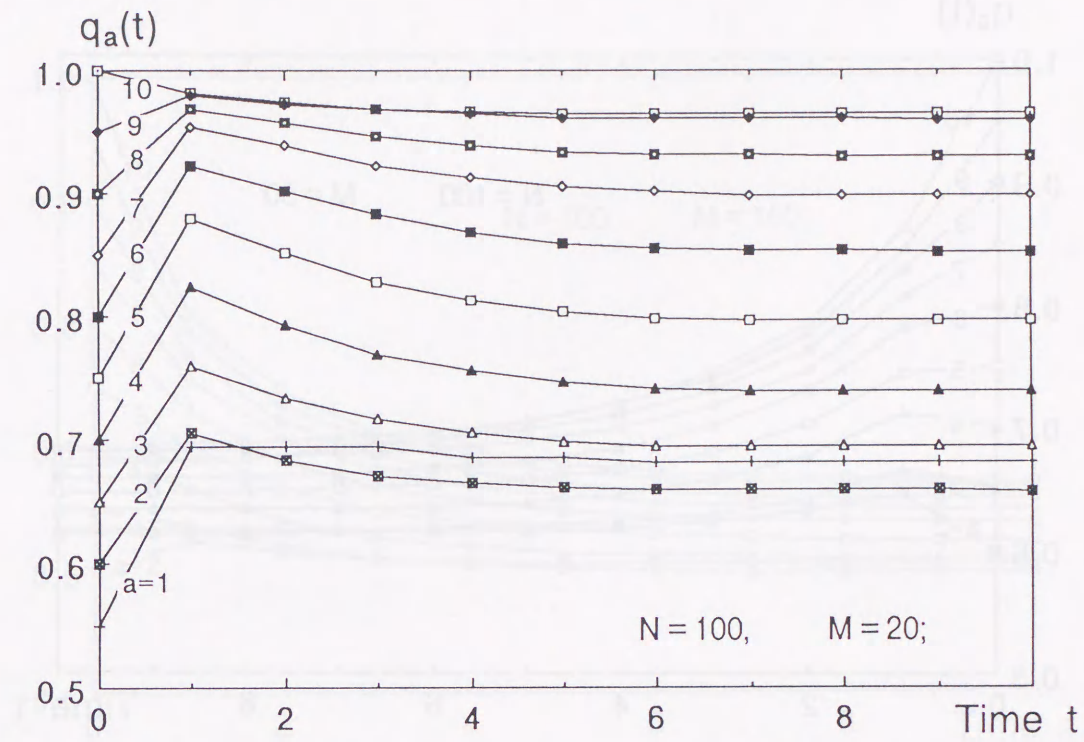



**Figure 4-4. b:** Statistical value on similar probability in Hopfiled model.

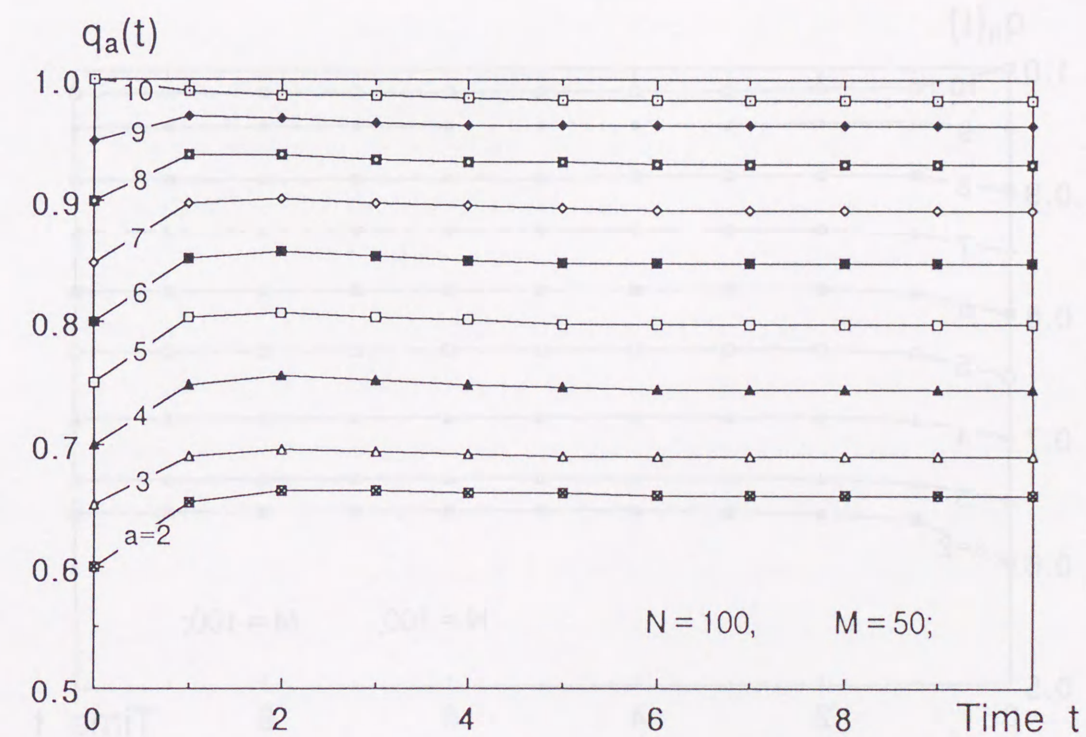



**Figure 4-4. c:** Statistical value on similar probability in ODAM model.

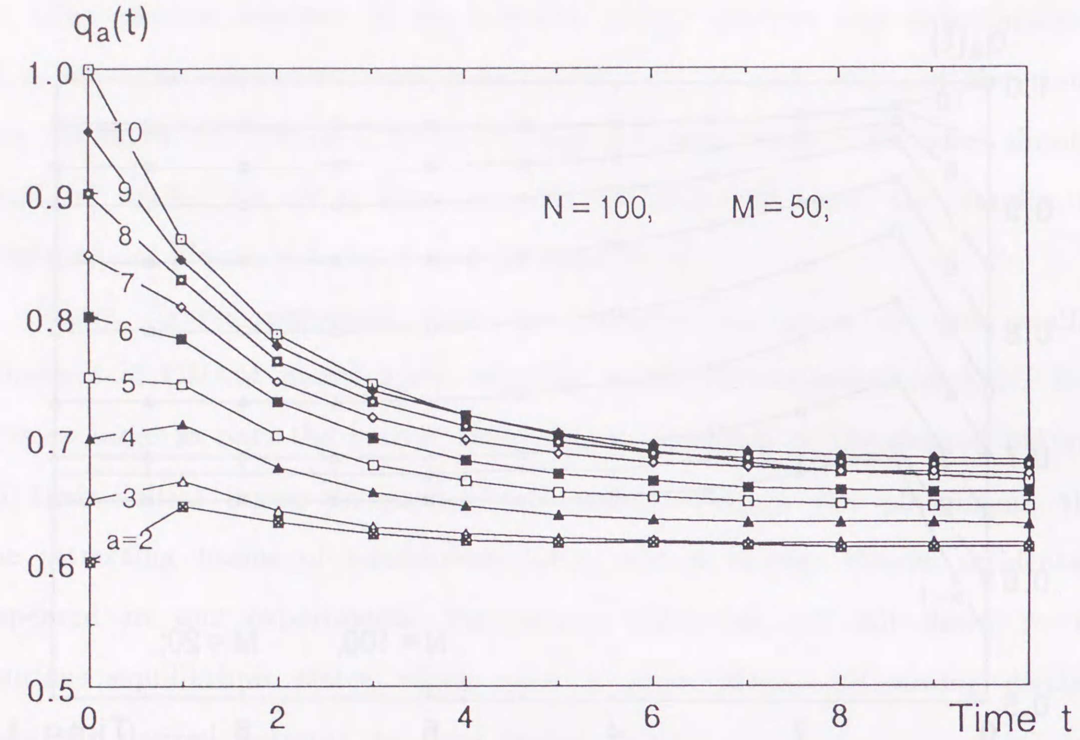



**Figure 4-4.** d: Statistical value on similar probability in Hopfiled model.

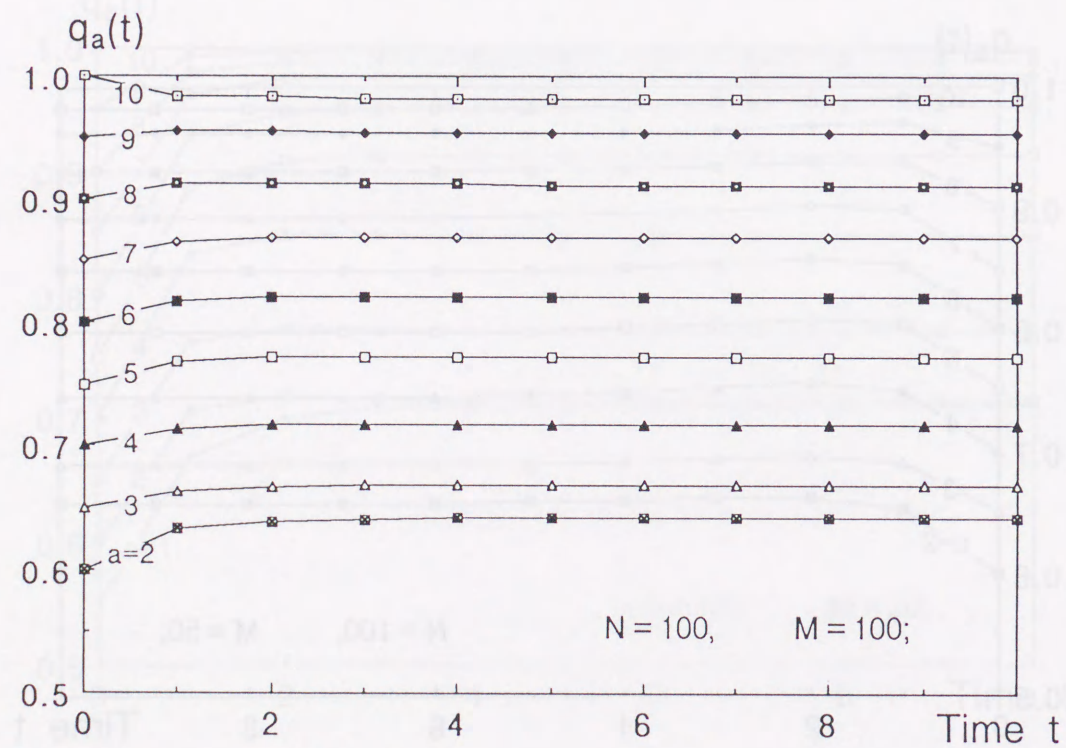



**Figure 4-4.** e: Statistical value on similar probability in ODAM model.



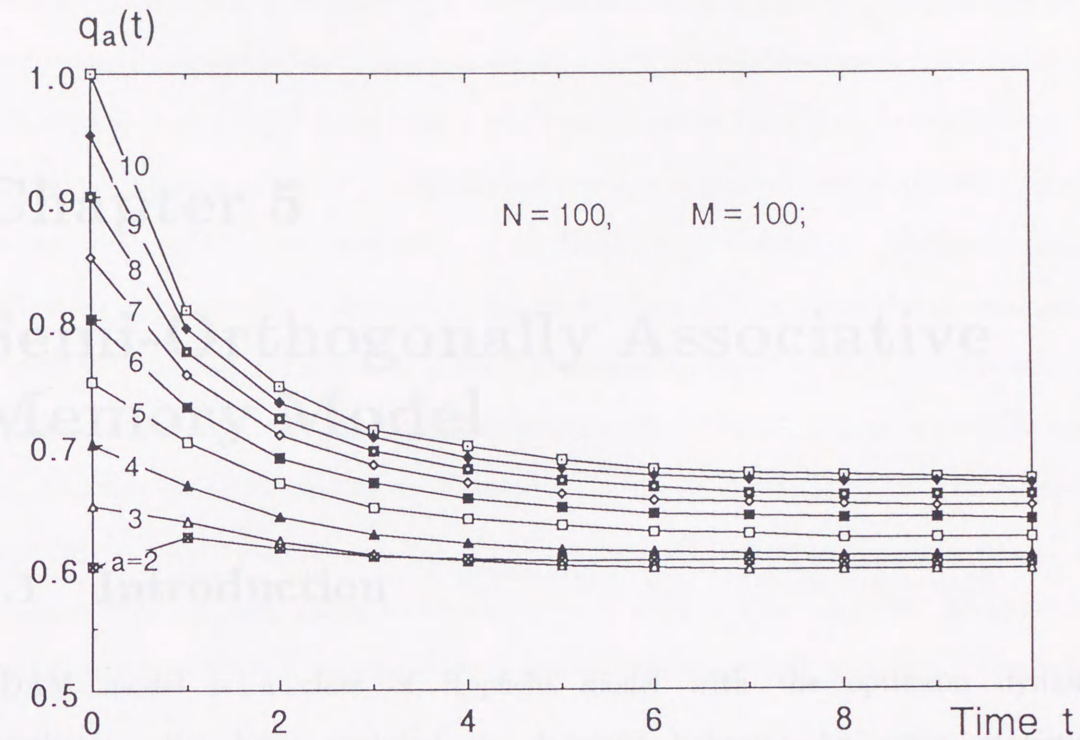



**Figure 4-4.** f: Statistical value on similar probability in Hopfiled model.

# Chapter 5

# Semi-Orthogonally Associative Memory Model

## 5.1 Introduction

ODAM model is a class of Hopfield model with the optimum dynamic threshold. We have analyzed its dynamic behavior by using statistical neurodynamics in Chapter 4 and confirmed the theoretical analysis results by simulation experiments. The dynamic behavior of ODAM model is better and its memory capacity is larger than those of Hopfield model. Furthermore, the phenomena of equilibrium states with strange attracting basins scarcely occurred in our experiments on this model. It implies that introducing the optimum dynamic threshold to Hopfield model for improving the phenomena is effective.

However, Like in Hopfield model, the similar probability in ODAM model does not tend to one, either. It is often occurred that the output series in the recalling processes converge to the points which are close to the desired patterns on their initial inputs but not the desired patterns. That is, there exist a lot of equilibrium states in the near neighborhood of the learned patterns in ODAM model. This is due to the training algorithm for constructing the connection weight matrix in this model.

The connection weight matrix in ODAM model, like in Hopfield model, is constructed by the summation of auto-correlation product of the learned patterns, too. The training algorithm constructing the connection weight matrix by the summation of auto-correlation product of the learned patterns is based on the Hebbian learning law. But, the correlation product of the learned patterns leads to that the error terms $(\theta_1(t),\cdots,\theta_N(t))$ in the recalling processes, which are defined in equation (3.13), depend on the correlations among the learned patterns and their values increase monotonously with the correlations. In fact, if learned pattern $A^{(\eta)}$ is the desired pattern on initial input $\mathbf{X}(0)$, we can rewrite the output $\mathbf{X}(t)$ $(t=0,1,\cdots)$ in the recalling processes as the summation of $A^{(\eta)}$ and an error vector $2\mathbf{Z}(t)$ as follows.

$$\mathbf{X}(t) = A^{(\eta)} + 2\mathbf{Z}(t), \qquad (t=0,1,\cdots) \tag{5.1}$$

where $\mathbf{Z}(t) \in \{-1,0,1\}^N$ and $\mathrm{P}\{z_i(t)\neq 0\} = 1-p(t)$ $(1\le i\le N)$. Then, substituting this equation into the error terms in the recalling processes, we obtain

$$\begin{aligned}(\theta_1(t),\cdots,\theta_N(t)) &= \sum_{1\le\xi\le m;\xi\neq\eta} \mathbf{X}(t)A^{(\xi)^T}A^{(\xi)} \\ &= \sum_{1\le\xi\le m;\xi\neq\eta} A^{(\eta)}A^{(\xi)^T}A^{(\xi)} + 2\sum_{1\le\xi\le m;\xi\neq\eta} \mathbf{Z}(t)A^{(\xi)^T}A^{(\xi)}. \end{aligned}\tag{5.2}$$

The first term in the right hand side of above equation represents the correlations among the learned patterns. It is not equal to zero except for the learned patterns are the orthogonal vectors in space $\mathbf{R}^N$. Accordingly, the error terms in the recalling processes are dependent on the correlations among the learned patterns and their values increase monotonously with the correlations. Furthermore, the first term usually is the main term in $\theta_i(t)$ $(1\le i\le N)$ because $\mathrm{P}\{z_i(t)\neq 0\} = 1-p(t)$ is small when the similar probability $p(t)\approx 1$.

It is a special example and no general sense that the learned patterns are considered to be mutually orthogonal. The condition that requires the learned patterns being strictly mutually orthogonal is too stringent to be applied,



because it can not be always satisfied in the real world. There always exist some inner relations among things in the real world and hence the patterns which indicate these things can not be strictly orthogonal each other. In general, the learned patterns in an associative memory model should not be assumed to satisfy any condition unless a specific task is set to this model.

However, this special example can be applied to the generalized associative memory model, because there is the characteristic layer in this model and the layer is completely independent of the concrete learned patterns. Even though the characteristic parameter of this model: the number of the characteristic sites, is also independent of the concrete learned patterns. Consequently, we can use this special example to form a training algorithm for constructing the connection weight matrix in the generalized associative memory model.

## 5.2 Semi-Orthogonal Training

The correlations among the learned patterns can not be directly improved but we can improve the error terms in the recalling processes by constructing effective connection weight matrix. In other words, we can look for more effective training algorithm for constructing the matrix.

Unlike in Hopfield model where the connection weight matrix is required to be symmetric and so on, there are no any restrictive conditions on the connection weight matrix in the generalized associative memory model. Further, the characteristic parameter in this model can be arbitrarily determined by the training algorithm. Therefore, we can applied the orthogonal vector in set $\mathbf{R}^n$ to the generalized associative memory model for constructing its connection weight matrix by the following algorithm which is referred to as semi-orthogonal training.

Let $A^{(1)},\cdots,A^{(m)}$ $(\in \mathbf{U}^N)$ be arbitrary learned patterns, and $O^{(1)},\cdots,O^{(m)}$

($\in \mathrm{U}^n$) be orthogonal vectors in space $\mathbf{R}^n$, i.e.,

$$O^{(\xi)}O^{(\eta)T} = \begin{cases} 0 & \xi \neq \eta \\ n & \xi = \eta \end{cases} \qquad (1 \leq \xi,\ \eta \leq m) \tag{5.3}$$

Then, the connection weight matrix in the generalized associative memory model can be defined as

$$\mathbf{W} = \sum_{\xi=1}^{m} A^{(\xi)^T} O^{(\xi)} \tag{5.4}$$

In other forms, the element of connection weight matrix is

$$\mathrm{w}_{ij} = \sum_{\xi=1}^{m} A_i^{(\xi)} O_j^{(\xi)} \qquad (i = 1, \cdots, N;\ j = 1, \cdots, n) \tag{5.5}$$

where $N$ and $m$ are numbers of neurons and learned patterns, respectively. Parameter $n$, which is called characteristic parameter of this model, is the dimension of middle states in the recalling processes of this model.

$$\mathbf{Y}(t) = Sgn[\mathbf{X}(t)\mathbf{W}] \tag{5.6}$$

Here, the orthogonal vector $O^{(\xi)}$ ($\in \mathrm{U}^n$) is referred to as the characteristic pattern corresponding to learned pattern $A^{(\xi)}$ ($\xi = 1, \cdots, m$) and set $\mathrm{U}^n$ is referred to as the characteristic space of this model. We refer to the generalized associative memory model with the connection weight matrix which is constructed with the semi-orthogonal training algorithm as Semi-orthogonally Associative Memory (SAM) model.

In general, $m$ orthogonal vectors may not exist in $\mathrm{U}^n$ for arbitrary positive integer $n$. But, if $n = 2^t$ ($t$ : positive integer) and $m \leq n$, $n$ orthogonal vectors in $\mathrm{U}^n$ can be simply given as following Example 5.1. Accordingly, in the next discussion, we only consider the characteristic parameter $n = 2^t$ ($t$ : positive integer) and $n \geq m$ so that the semi-orthogonal training algorithm can be efficiently applied to decide the connection weight matrix in the generalized associative memory model.



**Example 5.1:** In set $\mathrm{U}^2$, $n = 2$ row vectors of the matrix

$$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is one of the maximum orthogonal vectors set in $\mathbf{R}^2$. Generally, in set $\mathrm{U}^{2^t}$ ($t = 2, 3, \cdots$), $n = 2^t$ row vectors of the matrix

$$M_t = \begin{bmatrix} M_{t-1} & M_{t-1} \\ M_{t-1} & -M_{t-1} \end{bmatrix}. \tag{5.7}$$

is one of the maximum orthogonal vectors set in $\mathbf{R}^{2^t}$.

As we have seen, the semi-orthogonal training algorithm is based on the Hebbian learning law, too. But the connection weight matrix shown in equation (5.4) is not the summation of auto-correlation product of the learned patterns, and not the summation of hetero-correlation product between the learned patterns in different learned pattern set, either. The matrix is constructed by the summation of hetero-correlation product between the learned patterns and the characteristic patterns. These characteristic patterns are strictly mutually orthogonal and completely independent of the concrete learned patterns.

It can be considered as a self-organizing process[105],[106] in a sense that arbitrary learned patterns correspond to the strictly orthogonal characteristic patterns. In this self-organization, the learned patterns, which may be mutually dependent, are characterized as the patterns which are strictly orthogonal each other. Although this self-organization may be not optimal, it can partly put the correlations among the learned patterns out, indeed. We shall confirm this fact in the sequent sections of this chapter.

## 5.3 Dynamic Behavior of SAM Model

Like in the conventional associative memory models, we can use statistical neurodynamics to discuss the dynamic behavior of SAM model. Namely, using the similar probability defined in equation (3.9) to discuss the convergence of output series in the recalling processes of SAM model and derive the memory capacity and the information storage capacity per connection weight in this model.

### 5.3.1 Similar Probability in SAM Model

For arbitrary initial input $V = \mathbf{X}(0)$, by the definition of similar probability and the assumptions given in Chapter 3, we obtain that the similar probability at time $t$ in the recalling processes of SAM model is:

$$p_s(t+1) = P\left\{x_j(t+1) = A_j^{(\eta)}\right\} \qquad (1 \le j \le N) \tag{5.8}$$

Clearly, like in Hopfield model, this probability is independent of the subscript $j$, too. Next, we shall estimate it.

Since $O^{(\eta)}$ is the characteristic pattern corresponding to learned pattern $A^{(\eta)}$ and $A^{(\eta)}$ is assumed to be the desired pattern on arbitrary initial input $V = \mathbf{X}(0)$, then the middle state at time $t$ in the recalling processes can be rewritten as

$$\mathbf{Y}(t) = O^{(\eta)} - 2\beta(t), \qquad (t = 0, 1, \cdots) \tag{5.9}$$

where $\beta(t)(\in \{-1, 0, 1\}^n)$ is the error vector in $\mathbf{Y}(t)$ and

$$\begin{aligned} P\{\beta_i(t) \neq 0\} &= 1 - P\left\{y_i(t) = O_i^{(\eta)}\right\} \\ &= 1 - q(t) \qquad (i = 1, \cdots, n;\ t = 0, 1, \cdots) \end{aligned} \tag{5.10}$$



Here, probability $q(t)$ is

$$q(t) = P\left\{y_i(t) = O_i^{(\eta)}\right\} \tag{5.11}$$

Like the similar probability $p(t)$, probability $q(t)$ is also independent of the subscript $i$.

Thus, the similar probability

$$\begin{aligned} p_s(t+1) &= \mathbf{P}\left\{x_j(t+1) = A_j^{(\eta)}\right\} \\ &= \mathbf{P}\left\{h_j(t+1)A_j^{(\eta)} \ge 0\right\} \\ &= \mathbf{P}\left\{\mathbf{Y}(t)O^{(\eta)^T} - \sum_{\xi=1;\xi\neq\eta}^{m} [O^{(\eta)} - 2\beta(t)]O^{(\xi)^T} A_j^{(\xi)} A_j^{(\eta)} \ge 0\right\} \\ &= \mathbf{P}\left\{\sum_{\xi=1;\xi\neq\eta}^{m} \beta(t)O^{(\xi)^T} A_j^{(\xi)} A_j^{(\eta)} \le \frac{\mathbf{Y}(t)O^{(\eta)^T}}{2}\right\} \end{aligned} \tag{5.12}$$

Like in Chapter 3, here, we only consider the case $\mathbf{Y}(t)O^{(\eta)^T} \ge 0$, i.e., $p_s(t) \ge 1/2$. In the recalling processes, accurately estimating $\mathbf{Y}(t)O^{(\eta)^T}$ is difficult for every initial input. But, the expectation of $\mathbf{Y}(t)O^{(\eta)^T}$ is

$$E\left[\mathbf{Y}(t)O^{(1)^T}\right] = [2q(t) - 1]n,$$

and we can use it as an estimation of $\mathbf{Y}(t)O^{(\eta)^T}$. Substituting the expectation into equation (5.12) instead of term $\mathbf{Y}(t)O^{(\eta)^T}$. Then, if the dependence of error vector $\beta(t)$ on learned patterns $A^{(\xi)}$ and characteristic patterns $O^{(\xi)}$ $(1 \le \xi \le m,\ \xi \neq \eta)$ are neglected, the error terms

$$N_j(t) = \sum_{\xi=1\ \xi\neq\eta}^{m} \beta(t)O^{(\xi)^T} A_j^{(\xi)} A_j^{(\eta)} \qquad (1 \le j \le N) \tag{5.13}$$

are random variables and subject to an identical distribution. By the De Moivre-Laplace limit theorem shown in section 3.1, we have the distribution

**Table 5-1:** Similar probability in SAM model and Hopfield model.

| | m = 20; N = 100 | | m = 50; N = 100 | |
|---|---|---|---|---|
| | Hop. Model | SAM Model | Hop. Model | SAM Model |
| t = 1 | 0.672 | 0.760 | 0.612 | 0.613 |
| t = 2 | 0.761 | 0.996 | 0.625 | 0.628 |
| t = 3 | 0.876 | — | 0.640 | 0.647 |
| t = 4 | 0.959 | — | 0.655 | 0.672 |
| t = 5 | 0.979 | — | 0.670 | 0.702 |
| t = 6 | 0.982 | — | 0.684 | 0.743 |
| t = 7 | — | — | 0.699 | 0.796 |
| t = 8 | — | — | 0.713 | 0.859 |
| t = 9 | — | — | 0.726 | 0.920 |
| $P_{m,N}$ | 0.98310 | 1.00000 | 0.81530 | 0.99240 |

is normal distribution $N(0,\ [1-q(t)nm])$ when the number of neurons $N$ is sufficiently large. Therefore, the similar probability in the recalling processes of SAM model is

$$p_s(t+1) \approx \Phi\left(\frac{1}{2\nu}\frac{2q(t)-1}{\sqrt{1-q(t)}}\right) \tag{5.14}$$

where $\Phi(\cdot)$ is the standardized normal distribution function, parameter $\nu$ is defined as follows.

$$\nu = \sqrt{\frac{m}{n}} \tag{5.15}$$



and the probability $q(t)$ is defined in equation (5.11). as calculating similar probability $p_s(t)$, we can deduce the probability

$$q(t) = \Phi\left(\frac{2p_s(t)-1}{\sqrt{m/N}}\right). \tag{5.16}$$

Table 5-1 show the numerical solution of similar probability in the recalling processes of Hopfield model and SAM model. Here, the neuron number is $N = 100$, the number of learned patterns are $m = 20$ and 50 (i.e., the learned ratio $m/N = \lambda^2 = 0.2,\ 0.5$), respectively, and the characteristic parameter of SAM model is $n = 128$. The initial similar probability is assumed to be $p(0) = 0.6$ in both model. From this table we can see that the increasing rate and the limit of similar probability in the recalling processes of SAM model are much larger than those in Hopfield model. This implies that the dynamic behavior of SAM model is much better.

### 5.3.2 Converging Index of SAM model

Next, we shall use the similar probability to discuss the convergent properties of SAM model.

In equation (5.16), probability $q(t)$ is an increasing function of $p_s(t)$ and $q(t) \approx p_s(t)$. Then, we can use $p_s(t)$ as an approximation to $q(t)$. For simplicity, substituting $p_s(t)$ into the similar probability $p_s(t+1)$ instead of $q(t)$ and obtain an approximate expression

$$p_s(t+1) \approx \Phi\left(\frac{1}{2\nu}\frac{2p_s(t)-1}{\sqrt{1-p_s(t)}}\right) \tag{5.17}$$

Hence, in order to discuss the convergent properties and the relationships between the characteristic parameter and the dynamic behavior of SAM model by means of similar probability, we can consider the function

$$y(\nu, x) = \Phi\left(\frac{1}{2\nu}\frac{2x-1}{\sqrt{1-x}}\right) \qquad (0 < \nu \le 1;\ \ 0.5 \le x < 1) \tag{5.18}$$

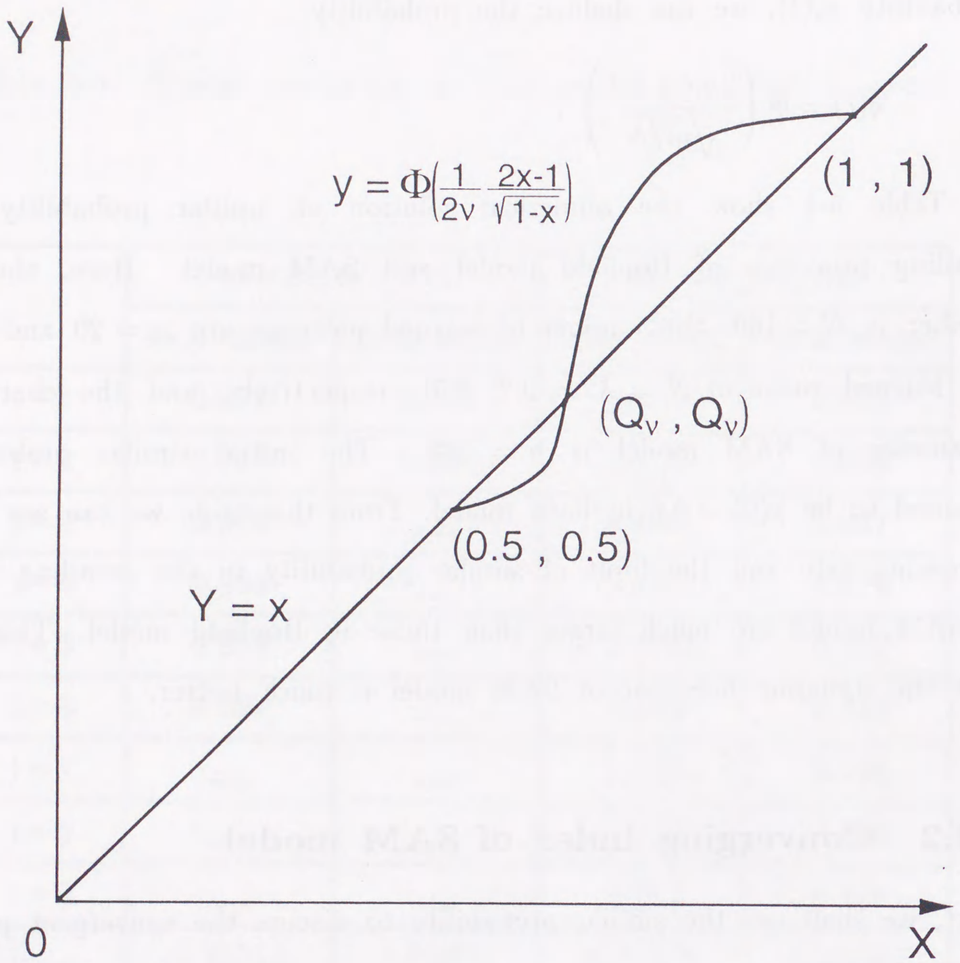



**Figure 5-1:** Image of function $y = \Phi(\frac{1}{2\nu}\frac{2x-1}{\sqrt{1-x}})$.

The image of function $y(\nu, x)$ is shown in Figure 5-1. In accordance with Remark 3.2, for any given $\nu$ ($\in (0,1]$), to guarantee that the output series in the recalling processes of SAM model approach to the desired pattern on their initial input in probabilistic sense, the following equation must be true.

$$y(\nu, x) - x \geq 0 \qquad (0.5 < x < 1) \tag{5.19}$$

However, this equation is not always true for some parameter $\nu$ ($\in (0,1]$).



Demonstrating this fact is not difficult.

Considering the derivative function

$$[y(\nu, x) - x]'_x = \frac{1}{4\nu\sqrt{2\pi}}\frac{3-2x}{(1-x)^{3/2}}e^{-\frac{1}{8\nu^2}\frac{(2x-1)^2}{1-x}} - 1$$

at point $x = 1/2$, when $\nu > 1/\sqrt{\pi}$, we have

$$[y(\nu, x) - x]'_x = \frac{1}{\nu\sqrt{\pi}} - 1 < 0, \tag{5.20}$$

and clearly

$$y(\nu, 0.5) - 0.5 = 0. \tag{5.21}$$

Hence, there exist $x'$ in the right hand neighborhood of point $x = 1/2$ such that

$$y(\nu, x') - x' < 0 \qquad (\nu > 1/\sqrt{\pi})$$

Namely, equation (5.19) is not always true when $\nu > 1/\sqrt{\pi}$.

Furthermore, by means of the derivative $[y(\nu, x) - x]'_x$ and the values of function $y(\nu, x)$ at point $x = 0.5$ and

$$\lim_{x \to 1^-} y(\nu, x) = 1 \tag{5.22}$$

discussing the monotone and continuous properties of function $y(\nu, x)$, we can obtain more general results on equation (5.19) as follows.

**1:** when $0 < \nu \leq 1/\sqrt{\pi}$, equation (5.19) is always true for all $x \in (0.5, 1)$, i.e.,

$$y(\nu, x) - x > 0 \qquad (0.5 < x < 1) \tag{5.23}$$

**2:** when $1/\sqrt{\pi} < \nu \leq 1$, equation

$$y(\nu, x) - x = 0 \tag{5.24}$$

has one and only one solution $Q_\nu$ $(0.5 < Q_\nu < 1)$ such that

$$y(\nu, x) - x < 0 \qquad (0.5 < x < Q_\nu)$$

$$y(\nu, x) - x > 0 \qquad (Q_\nu < x < 1)$$

That is, equation (5.19) is not always true in this case. Further, when $x = Q_\nu$, the derivative function

$$[y(\nu, x) - x]'_x > 0 \qquad (x = Q_\nu) \tag{5.25}$$

Consequently, let $Q_\nu = 0.5$ when $0 < \nu \leq 1/\sqrt{\pi}$, we obtain the following result.

**The convergence property of SAM model:** For any initial input $X(0) \in \mathrm{U}^N$, there exists a number $Q_\nu$ $(\geq 0.5)$ such that, when $p_s(0) > Q_\nu$, the output series in the recalling processes of SAM model converge to the desired pattern on $X(0)$ with probability one or it is not true.

♯

That is, there exist a parameter $Q_\nu$ in SAM model such that, for any initial input $X(0)$, when $p_s(0) > Q_\nu$, the desired pattern on the initial input can be recalled, and when $p_s(0) \leq Q_\nu$, this model is in a paralytic state and the desired pattern on the initial input can not be recalled in probabilistic sense. The parameter $Q_\nu$ is termed converging index of SAM model, which indicates the fundamental converging properties of SAM model. Table 5-2 displays numerical solutions of the converging index $Q_\nu$ when $\nu = 1/\sqrt{\pi}$, 0.75 and 1.

Converging index $Q_\nu$ is a function of characteristic parameter $n$ since $\nu = \sqrt{m/n}$. This suggests that, in the recalling processes, the anti-noise



**Table 5-2:** The converging index of SAM model.

| $\nu$ | $1/\sqrt{\pi}$ | 0.75 | 1 |
|---|---|---|---|
| $Q_\nu$ | 0.5 | 0.75 | 0.91 |

capability or error correcting capability of SAM model depend on the characteristics taken out in training processes. This property is similar to the recalling behavior and recognition behavior of human brains. For example, we can recognize the words, shown in Figure 5-2. a and b, as 'When' and 'Then', respectively, but we can not recognize the words shown in Figure 5-2. c and d if no an elaborate observation, as the noise is too much in these figures. However, if we have an elaborate observation and notice the characteristics that alphabet 'T' has a bar and 'W' does not, we can guess that the words may be 'When' and 'Then' shown in Figure 5-2. c and d, respectively. These characteristics of the alphabets 'T' and 'W' are known when we learn these alphabets. That is, the recalling capability or anti-noise capability of human being depend on the characteristics taken out in training processes.

In probabilistic sense, the converging index suggests that, for any non-spurious equilibrium state $A$ in state space $\mathrm{U}^N$ of SAM model, there are no spurious equilibrium states in the spherical neighborhood $O(A, 1 - Q_\nu)$

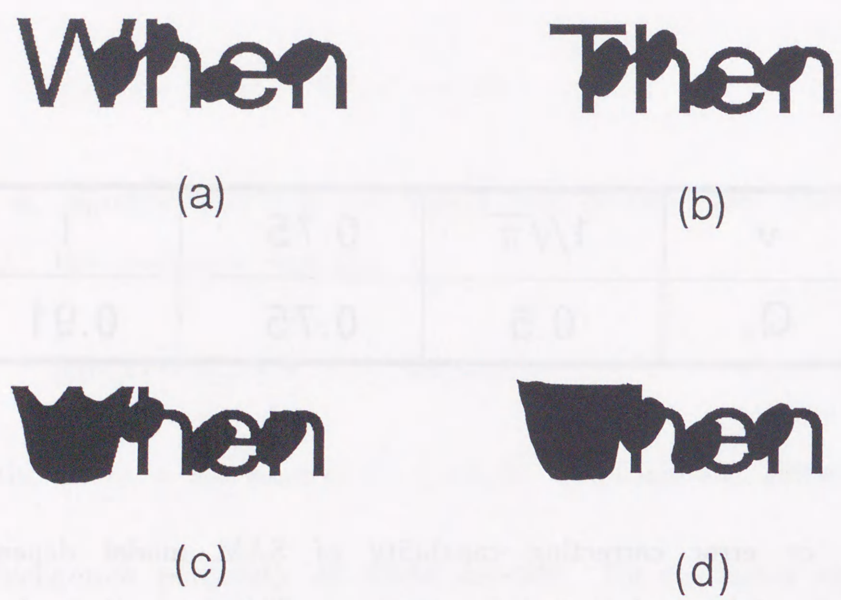
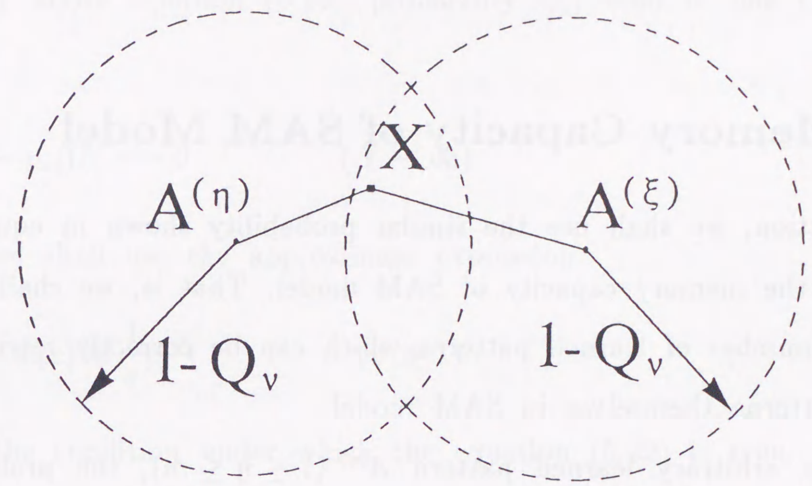



Figure 5-2: The example of recognition.

of pattern $A$. However, we must point out that the theoretical result is based on approximate statistical analysis and some assumptions which are not always true, say, the assumption that there is only one of desired pattern on any initial input in this model. Accordingly, it does not guarantee that the attracting basin of pattern $A$ contains the spherical neighborhood $O(A, 1-Q_\nu)$. For any $X \in O(A, 1-Q_\nu)$, we can not make sure that pattern $A$ is the desired pattern on $X$ by distance $D(X, A) \leq 1 - Q_\nu$. For instance, in Figure 5-3, let patterns $A^{(1)}$ and $A^{(2)}$ indicate two different non-spurious equilibrium states in SAM model and suppose $D(A^{(1)}, X) < D(A^{(2)}, X) < 1 - Q_\nu$, then



pattern $A^{(2)}$ is not the desired pattern on $X$ by the definition of desired pattern although $D(A^{(2)}, X) < 1 - Q_\nu$.




$$D(A^{(\eta)}, X) < D(A^{(\xi)}, X) \leq 1 - Q_\nu$$

Figure 5-3: The example of converging region.

Furthermore, when the assumption that there is only one desired pattern on an initial input is not satisfied, the theoretical analysis result is not true for this initial input. In this case, the spurious equilibrium states may occur in the spherical region $O(A, 1-Q_\nu)$ of non-spurious equilibrium state $A$. In general, the probability that this event occurs increases with the parameters $\lambda$ and $\nu$. But, when $\lambda$ and $\nu$ are small, this probability is very small and above theoretical analysis is reliable and significant.

The case that an initial input has multi-desired patterns in the recalling processes should not be neglected in the statistical analysis of associative memory model when the number of learned patterns is large relative to the number of neurons. For more accurately analyzing the convergent properties of SAM model, it is necessary and expected to study this case in detail.

## 5.4 Memory Capacity of SAM Model

In this section, we shall use the similar probability shown in equation (5.14) to discuss the memory capacity of SAM model. That is, we shall deduce the maximum number of learned patterns which can be correctly retrieved by the learned patterns themselves in SAM model.

For an arbitrary learned pattern $A^{(\eta)}$ $(1 \leq \eta \leq m)$, the probability that the pattern $A^{(\eta)}$ can be correctly retrieved by the pattern $A^{(\eta)}$ itself, namely, the probability that the pattern $A^{(\eta)}$ is stored at SAM model is

$$\begin{aligned} Q_{s1} &= P\{\mathbf{X}(1) = A^{(\eta)}\} \\ &= P\{x_1(1) = A_1^{(\eta)}, \cdots, x_N(1) = A_N^{(\eta)}\} \\ &\approx \prod_{i=1}^{N} P\{x_i(1) = A_i^{(\eta)}\} \end{aligned} \tag{5.26}$$

where $\mathbf{X}(1)$ $(\in U^N)$ is the output at time $t=1$ in the recalling processes on initial input $\mathbf{X}(0) = A^{(\eta)}$. Hence, $P\{x_i(1) = A_i^{(\eta)}\}$ $(1 \leq i \leq N)$ is the similar probability at time $t=1$ when the initial similar probability $p_s(0) = 1$. In accordance with equation (5.14), we have

$$\begin{aligned} p_{s1} &\triangleq P\{x_i(1) = A_i^{(\eta)}\} \qquad (1 \leq i \leq N) \\ &\approx \Phi\left(\frac{1}{2\nu}\frac{2\Phi(1/\lambda) - 1}{\sqrt{1-\Phi(1/\lambda)}}\right) \end{aligned} \tag{5.27}$$

Substituting it into equation (5.26) and using the Poisson approximation, we have



$$Q_{s1} \approx e^{-(1-p_{s1})N} \tag{5.28}$$

Therefore, in probabilistic sense, for arbitrary learned patterns $A^{(\eta)}$, it is correctly stored at SAM model if and only if probability $Q_{s1}$ tend to one as $N \to \infty$. By above equation (5.28), probability $Q_{s1}$ tend to one $(N \to \infty)$ if and only if

$$(1 - p_{s1})N \longrightarrow 0 \qquad (N \to \infty) \tag{5.29}$$

Next, we shall use the approximate expression

$$1 - \Phi(x) \approx \frac{1}{x}e^{-\frac{x^2}{2}} \qquad (x \gg 1) \tag{5.30}$$

to deduce the condition under which the equation (5.29) is true.

For $(1 - p_{s1})N$, making use of approximate expression (5.30), we obtain an approximate equality

$$(1 - p_{s1})N = \frac{N}{u}e^{-\frac{u^2}{2}}, \tag{5.31}$$

where

$$u = \frac{1}{2\nu}\frac{2\Phi(1/\lambda) - 1}{\sqrt{1-\Phi(1/\lambda)}} \tag{5.32}$$

Consequently, probability $Q_{s1}$ tend to one $(N \to \infty)$ if and only if

$$\frac{u^2}{2} + \ln u - \ln N \to +\infty \qquad (N \to \infty) \tag{5.33}$$

Clearly, this equation is true, when

$$\frac{u^2}{2} \geq \ln N \qquad (N \to \infty) \tag{5.34}$$

Substituting the variable $u$ defined in equation (5.32) into the above condition equation, and using the approximate expression (5.30) to its denominator term again and hence the above condition equation change as follows.

$$\frac{(2\Phi(1/\lambda)-1)^2}{8\nu^2\lambda}e^{\frac{1}{2\lambda^2}} \geq \ln N \qquad (N \to \infty)$$

Therefore, if

$$\frac{N}{m} = \frac{1}{\lambda^2} \geq 2(\ln\ln N + \ln\delta) \qquad (N \to \infty) \tag{5.35}$$

is true, the probability $Q_{s1}$ must tend to one $(N \to \infty)$, where the parameter $\delta$ is:

$$\delta = \frac{8\nu^2\lambda}{(2\Phi(1/\lambda)-1)^2} \tag{5.36}$$

Here, the parameter $\nu$ is not larger than one since $m \leq n$. Further, if we consider the parameter $\lambda \leq 1$, then the term $\ln\delta$ is small and can be neglected.

Consequently, in SAM model, when the neuron number $N$ is sufficiently large and the number of learned patterns

$$m \leq \frac{N}{2\ln\ln N}, \tag{5.37}$$

then, all the learned patterns are recoverable with probability one in SAM model. In other words, the memory capacity of SAM model is

$$m_c^s = \frac{N}{2\ln\ln N} \tag{5.38}$$

**REMARK 5.1:** Comparing the memory capacity of SAM model with Hopfield model and BAM model, we have

$$\frac{m_c^s}{N} = \frac{1}{2\ln\ln N}$$

is decreasing with the number of neurons and tends to zero when $N \longrightarrow \infty$. This property is similar to those in Hopfield model and BAM model. In Hopfield model,

$$\frac{m_c^h}{N} = \frac{1}{2\ln N} \longrightarrow 0 \qquad (N \longrightarrow \infty)$$



However, when the number of neurons is large, the memory capacity of SAM model is much larger than those of Hopfield model and BAM model. The larger the neuron number $N$ is, the larger the increasing times of the memory capacity is. Table 5-3 shows the numerical solution about the memory capacity of SAM model, BAM model and Hopfield model. For example, when $N = 10^3$, the memory capacity of SAM model is 3.6 times and 1.8 times those of Hopfield model and BAM model, respectively, and when $N = 10^6$, it is 5.4 times and 2.7 times, respectively.

**Table 5-3:** Numerical solutions of the memory capacity of SAM model, Hopfield model and BAM model.

| | Hopfield Model | BAM Model | SAM Model |
|---|---|---|---|
| Theoretical Estimating | $\frac{N}{2\ln N}$ | $\frac{N}{\ln N}$ | $\frac{N}{2\ln\ln N}$ |
| $N=10^2$ | 0.1086N | 0.2172N | 0.3274N |
| $N=10^3$ | 0.0724N | 0.1447N | 0.2587N |
| $N=10^4$ | 0.0543N | 0.1086N | 0.2369N |
| $N=10^5$ | 0.0434N | 0.0869N | 0.2046N |
| $N=10^6$ | 0.0362N | 0.0724N | 0.1904N |

**REMARK 5.2:** In general, the memory capacity of associative memory model is dependent on the number of connection weights. That is, the memory capacity of associative memory model depends on the raw number and column number of the connection weight matrix. In BAM model, if the numbers of neuron fields $A$ and $B$ are $N_1$ and $N_2$, respectively, its memory capacity is

$$m_c^b = \min\left\{\frac{N_1}{\ln N_1}, \frac{N_2}{\ln N_2}\right\}$$

By equation (5.38), but, the memory capacity of SAM model is hardly dependent on its characteristic parameter, ziv., the column number of connection weight matrix if the higher order infinitesimal $\ln\delta$ defined in equation (5.36) is ignored. This property implies that the information storage capacity per connection weight can be enlarged by determining the characteristic parameter is small as possible in SAM model.

Here, by means of the simulation experiments on computer, we shall examine these statistical analyses results on the memory capacity of SAM model.

We simulate the models with neuron numbers $N = 100$ and 1000 in our experiments. The learned patterns number in SAM model, BAM model and Hopfield model are all the same. i.e., the learned ratio is $m/N = 0.01, 0.02, \cdots, 0.60$ in both cases of $N = 100$ and $N = 1000$. The characteristic parameters of SAM model are $n = 2^7 = 128$ and $n = 2^{10} = 1024$ for the cases $N = 100$ and 1000, respectively. All of the learned patterns, which are produced by the random function in computer, are random and satisfy $P\{A_j^{(\xi)} = \pm 1\} = 1/2$. In the experiments on BAM model, we always assumed that the desired pattern on any initial input is in which of the neuron fields $A$ and $B$ has been known before the recalling processes start to do.

We respectively examined the rates which the learned patterns are



correctly stored in SAM model, BAM model and Hopfield model. The simulation tests are repeatedly done 200 for each learned pattern number in every model. In Figure 5-4, we show the mean values of the rates for all experiments in each case.

The experimental results are shown in Figure 5-4 a for $N = 100$. In this case, when $m/N$ is larger than $0.30 \approx m_c^s/N$ in SAM model, we can see that the rate which the learned patterns are correctly stored begin to decrease speedily with $m/N$ and tend to zero, but when $m/N < 0.30$, the rate is about one. And in the case of $N = 1000$, the results are shown in Figure 5-4 b, when $m/N$ is larger than $0.26 \approx m_c^s/N$ in SAM model, the rate begin to decrease speedily, but when $m/N < 0.26$, the rate is larger

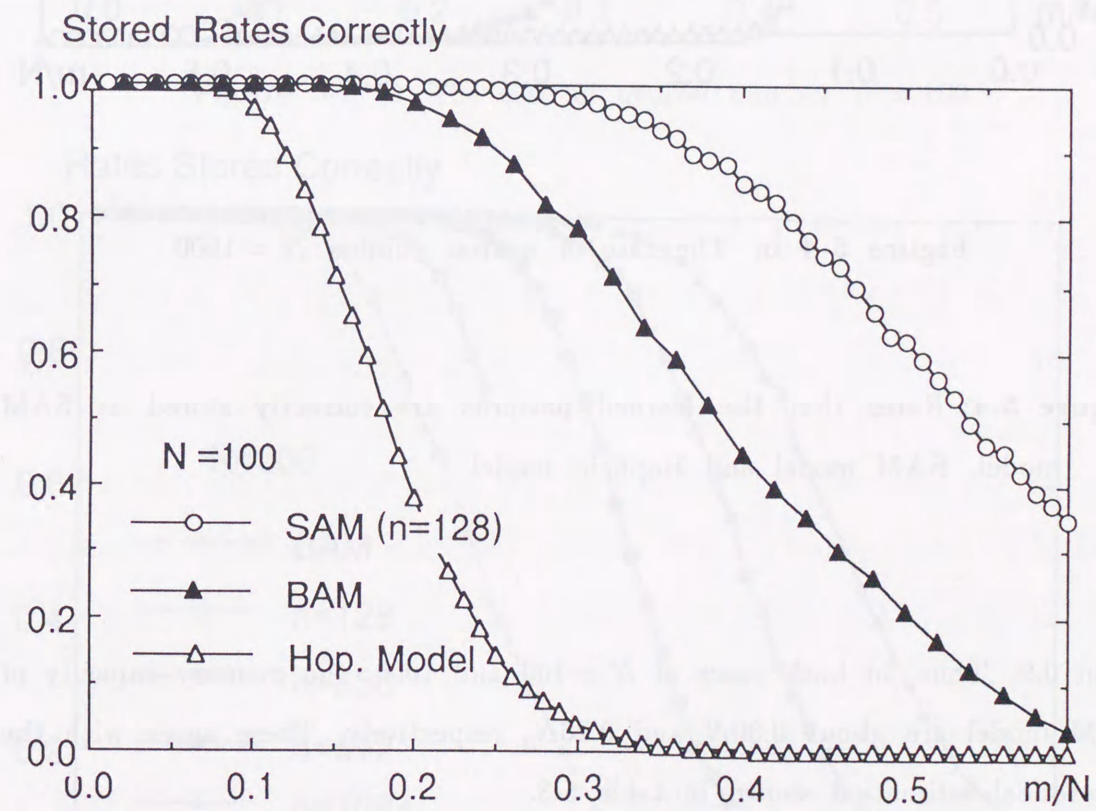



**Figure 5-4 a:** The case of neuron number $N = 100$

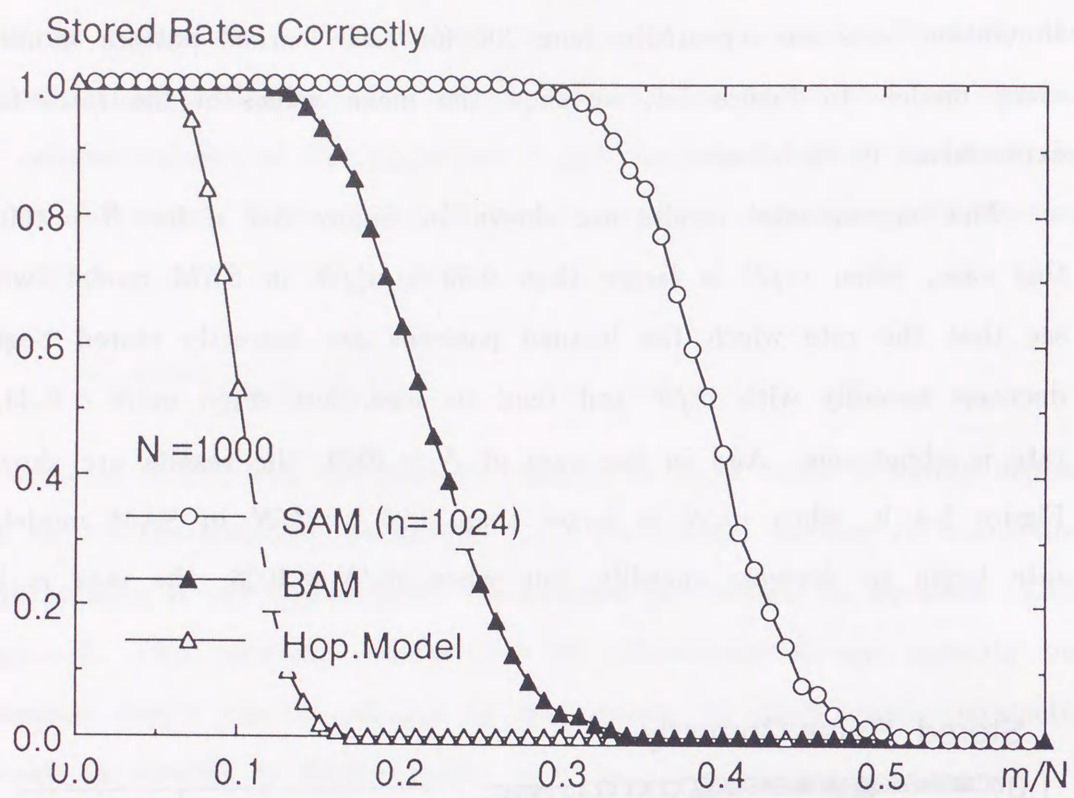



**Figure 5-4 b:** The case of neuron number $N = 1000$

**Figure 5-4:** Rates that the learned patterns are correctly stored at SAM model, BAM model and Hopfield model

than 0.9. Thus, in both cases of $N = 100$ and 1000, the memory capacity of SAM model are about $0.30N$ and $0.26N$, respectively. These agree with the theoretical estimation shown in table 5-3.

Further, comparing with BAM model and Hopfield model, the enlarged times of the memory capacity of SAM model agree with the theoretical



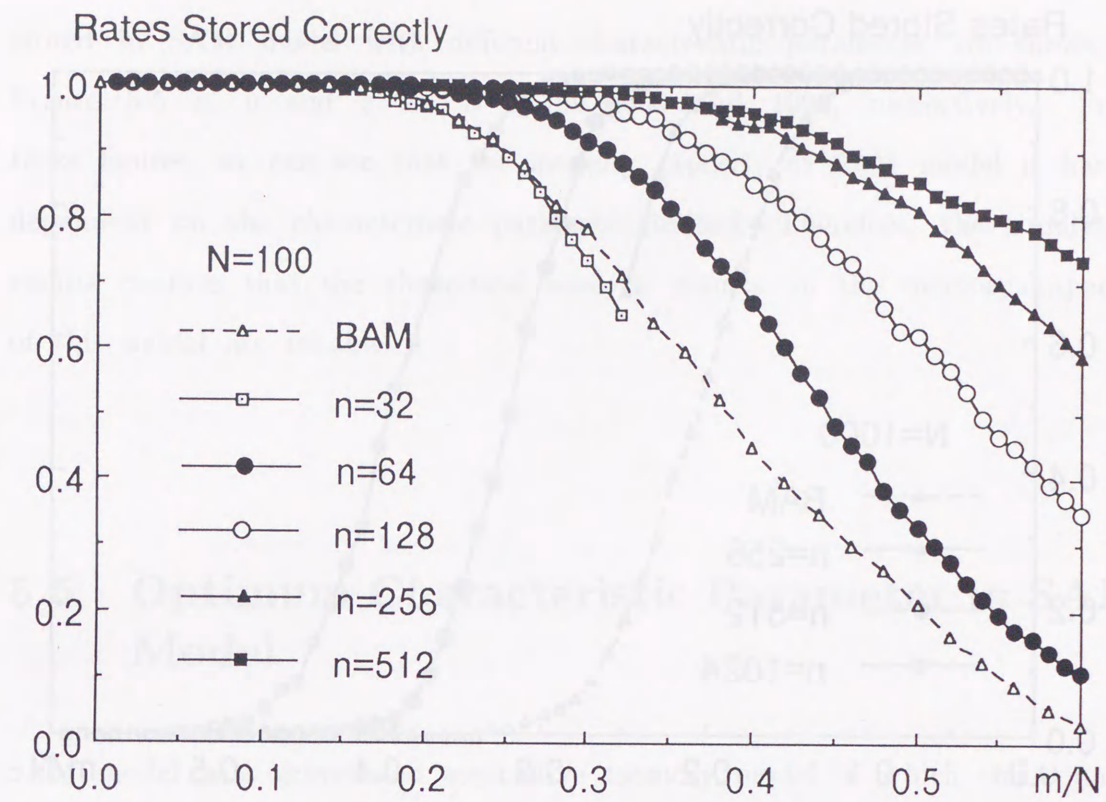



**Figure 5-5 a:** The case of neuron number $N = 100$

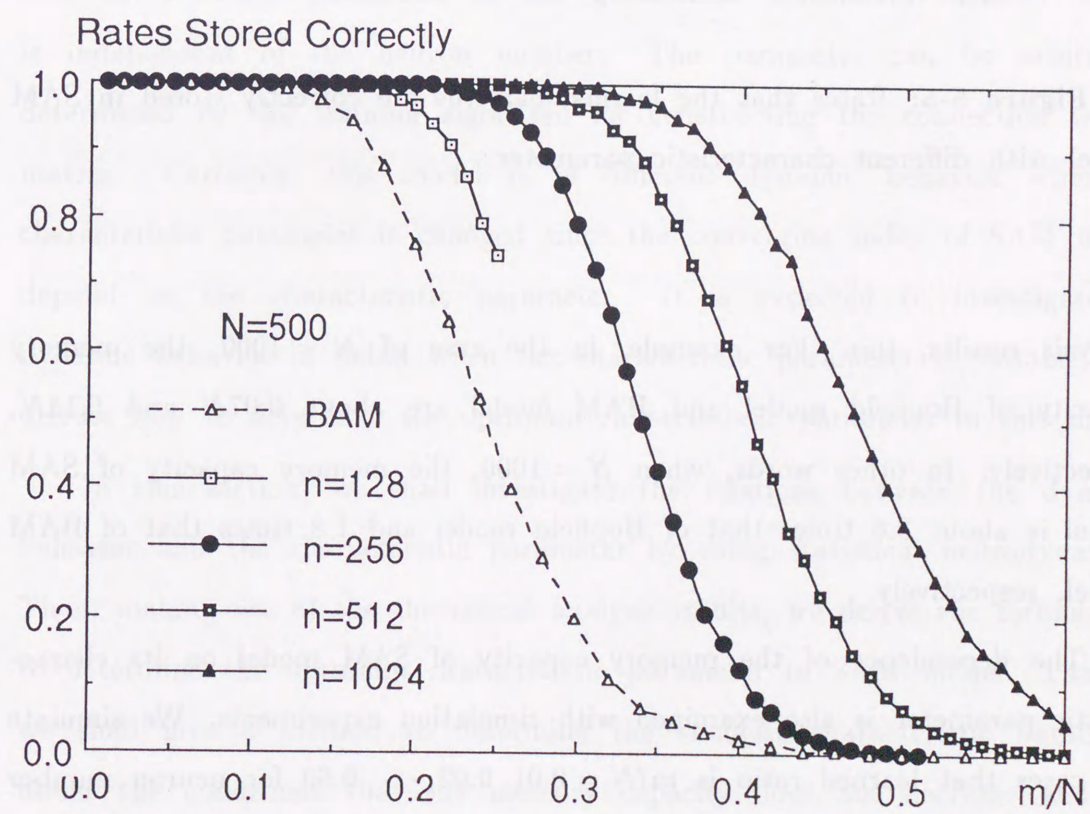



**Figure 5-5 b:** The case of neuron number $N = 500$

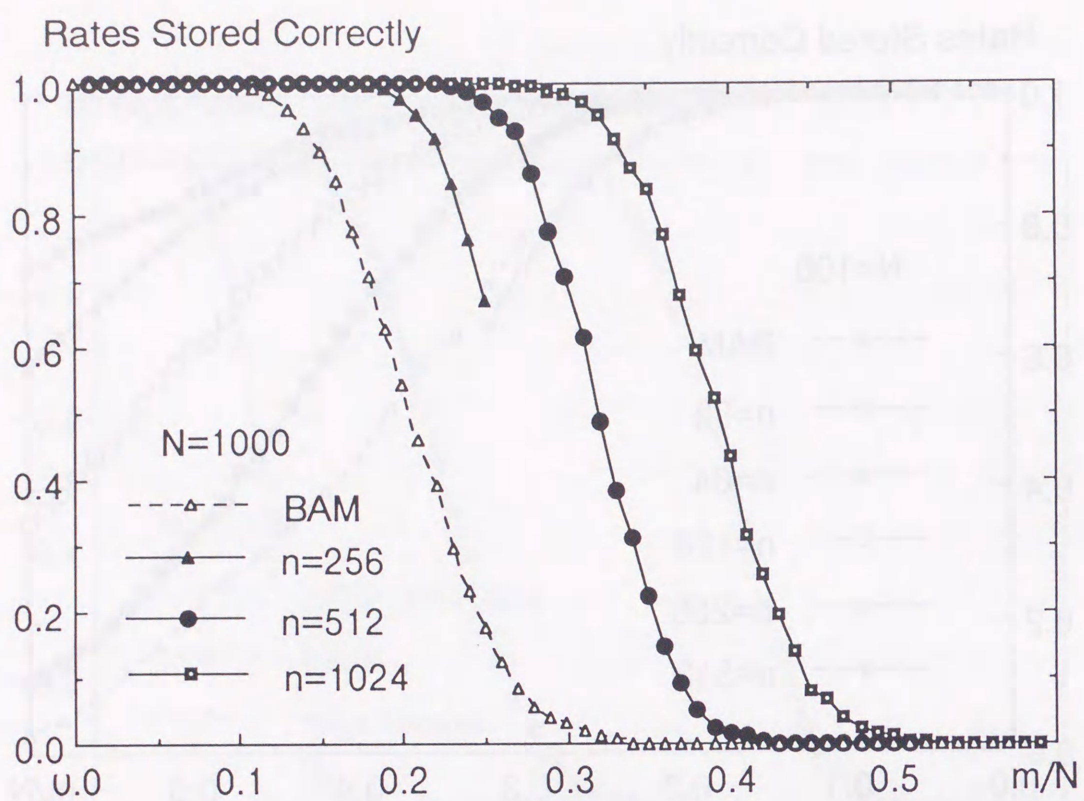



**Figure 5-5 c:** The case of neuron number $N = 1000$

**Figure 5-5:** Rates that the learned patterns are correctly stored in SAM model with different characteristic parameter.

analysis results, too. For example, in the case of $N = 1000$, the memory capacity of Hopfield model and BAM model are about $0.07N$ and $0.14N$, respectively. In other words, when $N = 1000$, the memory capacity of SAM model is about 3.6 times that of Hopfield model and 1.8 times that of BAM model, respectively.

The dependence of the memory capacity of SAM model on its characteristic parameter is also examined with simulation experiments. We simulate the cases that learned ratio is $m/N = 0.01, 0.02, \cdots, 0.60$ for neuron number $N = 100$, 500 and 1000. The rates which the learned patterns are correctly



stored in SAM model with different characteristic parameter are shown in Figure 5-5 a, b and c for $N = 100$, 500 and 1000, respectively. From these figures, we can see that the memory capacity of SAM model is hardly dependent on the characteristic parameter indeed. Therefore, the simulation results confirm that the theoretical analysis results on the memory capacity of this model are reliable.

## 5.5 Optimum Characteristic Parameter in SAM Model

SAM model is a generalized associative memory model in which the connection weight matrix is constructed by the semi-orthogonal training algorithm. The characteristic parameter in the generalized associative memory model is independent of the neuron number. The parameter can be arbitrarily determined by the training algorithm for constructing the connection weight matrix. Certainly, this model is of different dynamic behavior when its characteristic parameter is changed since the converging index of SAM model depend on the characteristic parameter. It is expected to investigate its dynamic behavior in detail when the characteristic parameter is variable, and discuss how to determine the optimum characteristic parameter in this model.

In this section, we shall investigate the relations between the dynamic behavior and the characteristic parameter by using statistical neurodynamics. Then, making use of the theoretical analysis results, we derive the formulation to determine the optimum characteristic parameter in SAM model. That is, we shall give a method to determine the smallest characteristic parameter under the conditions that the memory capacity does not decrease and the error correcting capability does not weaken. Last, we shall confirm the theoretical results by simulation experiments.

### 5.5.1 Optimum characteristic parameter

We have demonstrated that the memory capacity of SAM model is scarcely dependent on the characteristic parameter $n\ (\geq m)$ by the statistical analysis, and this fact is confirmed by the simulation experiments, shown in Figure 5-5. This properties suggests that, in SAM model, the influence of the characteristic parameter on the memory capacity can be ignored.

However, in the definition of this model, the size of connection weight matrix $\mathbf{W}$ is $N \times n$, viz., the number of connection weights depends on $n$ and rapidly increases with it. In an associative memory model, the information storage capacity per connection weight is defined as follows.

$$S_c = \frac{\text{bit number of stored information}}{\text{number of the connection weight}} \quad \text{(bits/weight)} \tag{5.39}$$

Here, the bit number of stored information is equal to the product of the dimension of output pattern and the memory capacity of this model. Therefore, for a given SAM model with $N$ neurons, the information storage capacity per connection weight is

$$S_c = \frac{Nm_c}{Nn} \quad \text{(bits/weight)} \tag{5.40}$$

Because the memory capacity of SAM model $m_c^s$ is hardly dependent on $n$, we thus have: for a given SAM model with $N$ neurons, the less the characteristic parameter $n$ is, the larger the information storage capacity per connection weight is.

But, the converging index in SAM model is dependent on the characteristic parameter. In other words, the size of the equilibrium state attracting basins depend on the characteristic parameter. Consequently, the characteristic parameter can not be arbitrarily determined so that the size of the equilibrium state attracting basins is not reduced.



These factors together implied that the optimum characteristic parameter in SAM model should be given. That is, the smallest characteristic parameter should be given under the condition which the size of the equilibrium state attracting basins is not reduced. We shall first investigate the relation between the characteristic parameter and the converging index in detail with similar probability $p_s(t)$ and then deduce the optimum characteristic parameter in SAM model.

By Remarks 3.2 and 3.3, in probabilistic sense, the desired pattern on any initial input can be recalled by a SAM model with $N$ neurons, when

$$p_s(t+1) - p_s(t) > 0 \qquad (\text{for all } t = 0, 1, \cdots;) \tag{5.41}$$

and

$$\lim_{t \to \infty} p_s(t) = 1 \tag{5.42}$$

otherwise it is not true.

In accordance with the property of function $y(\nu, x)$ given by equation (5.18) and the convergence properties of SAM model shown in section 5.3, we obtain that the equations (5.41) and (5.42) are always true if the converging index $Q_\nu$ is equal to 1/2. The converging index is a function of $\nu$ and it is not less than 1/2 for any $\nu \in (0, 1]$. Therefore, we can guarantee that the equations (5.41) and (5.42) are true by means of restricting parameter $\nu = \sqrt{m/n}$.

By equations (5.24) and (5.25), we have

$$Q'_\nu = -\frac{\partial y}{\partial \nu} \Big/ \left(\frac{\partial y}{\partial x} - 1\right)\Bigg|_{x=Q_\nu} > 0 \qquad (1/\sqrt{\pi} < \nu \leq 1), \tag{5.43}$$

i.e., the converging index is monotone increasing with $\nu (\in (1/\sqrt{\pi}, 1])$, and

$Q_\nu = 0.5$ when $0 < \nu \le 1/\sqrt{\pi}$. Then, in order to guarantee that equations (5.41) and (5.42) are true, it is necessary to require

$$n > \pi m. \tag{5.44}$$

Notice that the memory capacity of this model is $m_c = N/2\ln\ln N$. Consequently, for a given SAM model with $N$ neurons, we should determine the characteristic parameter $n_1 = 2^{\tau_1}$ as follows.

$$\tau_1 = [\log_2 \pi m_c] + 1 \tag{5.45}$$

where $[\cdot]$ indicates the integral function. The number $n_1 = 2^{\tau_1}$ is the optimum characteristic parameter of SAM model with $N$ neurons. Clearly, it satisfies

$$\pi m_c \le n \le 2\pi m_c \tag{5.46}$$

Substituting this equation into equation (5.40), we obtain

$$\frac{1}{\pi} \ge S_c \ge \frac{1}{2\pi} \qquad \text{(bits/weight)} \tag{5.47}$$

Comparing with the convertional associative memory models, this result suggests that the information storage capacity per connection weight of SAM model is much larger.

The above result is obtained under the condition that guarantees the converging index $Q_\nu = 1/2$. Clearly, this condition is a sufficient condition for the size of equilibrium state attracting basins not to be reduced. In general, this condition is too stringent and insignificant.

For any $X$, $Y \in \mathrm{U}^N$, the standard effective Hamming distance between $X$ and $Y$ always satisfy

$$D(X,Y) \le \frac{1}{2} \qquad (\text{for all } X, Y \in \mathrm{U}^N)$$



Hence, for arbitrary learned patterns set $A^{(\xi)}$ $(\xi = 1, \cdots, m)$ and any number $d > 1/4$, then there must exist the patterns in $O(A^{(\xi)}, d)\backslash O(A^{(\xi)}, 1/4)$ which have multi-desired patterns when $m > 1$. Further, the number of this kind of patterns speedily increases with the number of learned patterns

**Table 5-4:** Numerical solutions of the optimum and generalized optimum characteristics parameters.

| N | 1/2lnlnN | $\tau_1$ | $\tau$ |
|---|---|---|---|
| 100 | 0.3272 | 7 | 7 |
| 200 | 0.3000 | 8 | 7 |
| 500 | 0.2738 | 9 | 9 |
| 1000 | 0.2587 | 10 | 10 |
| 2000 | 0.2478 | 11 | 10 |
| 5000 | 0.2332 | 12 | 12 |
| 10000 | 0.2251 | 13 | 13 |

$m$ in this model. When this kind of patterns are inputted to SAM model, the recalling are usually fail and above statistical analysis does not hold, either. Accordingly, restricting the converging index $Q_\nu = 1/2$ is too stringent and insignificant. Conversely, the number of this kind of patterns are considerable small in $O(A^{(\xi)}, 1/4)$, if the learned patterns are approximately subject to the uniform distribution in $\mathrm{U}^N$ and the number of the learned patterns is small.

Therefore, requiring $Q_\nu \le 3/4$ is nearly sufficient and reasonable when the optimum characteristic parameter is deduced. By table 5-2, we have $Q_\nu < 3/4$ when $\nu = \sqrt{m/n} < 3/4$. In other words, $Q_\nu < 3/4$ when

$$n > \frac{m}{(3/4)^2}$$

Then, we obtain a generalized optimum characteristic parameter $n' = 2^{\tau}$ where $\tau$ is given as follows,

$$\tau = [\log_2 2m_c] + 1 \tag{5.48}$$

In this case, the generalized optimum characteristic parameter $n'$ satisfies

$$2m_c \leq n \leq 4m_c \tag{5.49}$$

and the information storage capacity per connection weight in SAM model is

$$\frac{1}{2} \geq S_c \geq \frac{1}{4} \qquad \text{(bits/weight)} \tag{5.50}$$

That is, the information storage capacity per connection weight is enlarged. But the cost is that the recalling outputs do not converge to the desired pattern on their initial input when the distance between the initial input and their desired pattern is larger than $1 - Q_{\nu} \geq 1/4$. However, from above discussion, we have seen the cost does not really exist. Therefore, it is better selecting the generalized optimum characteristic parameter in SAM model. The numerical solutions of the optimum and generalized optimum characteristics parameters are shown in Table 5-4.

### 5.5.2 Computer Simulation

Here, we shall do simulation experiments for confirming the theoretical analysis results. In our simulations, all of the learned patterns are generated by the random function in computer. The simulation experiments are done



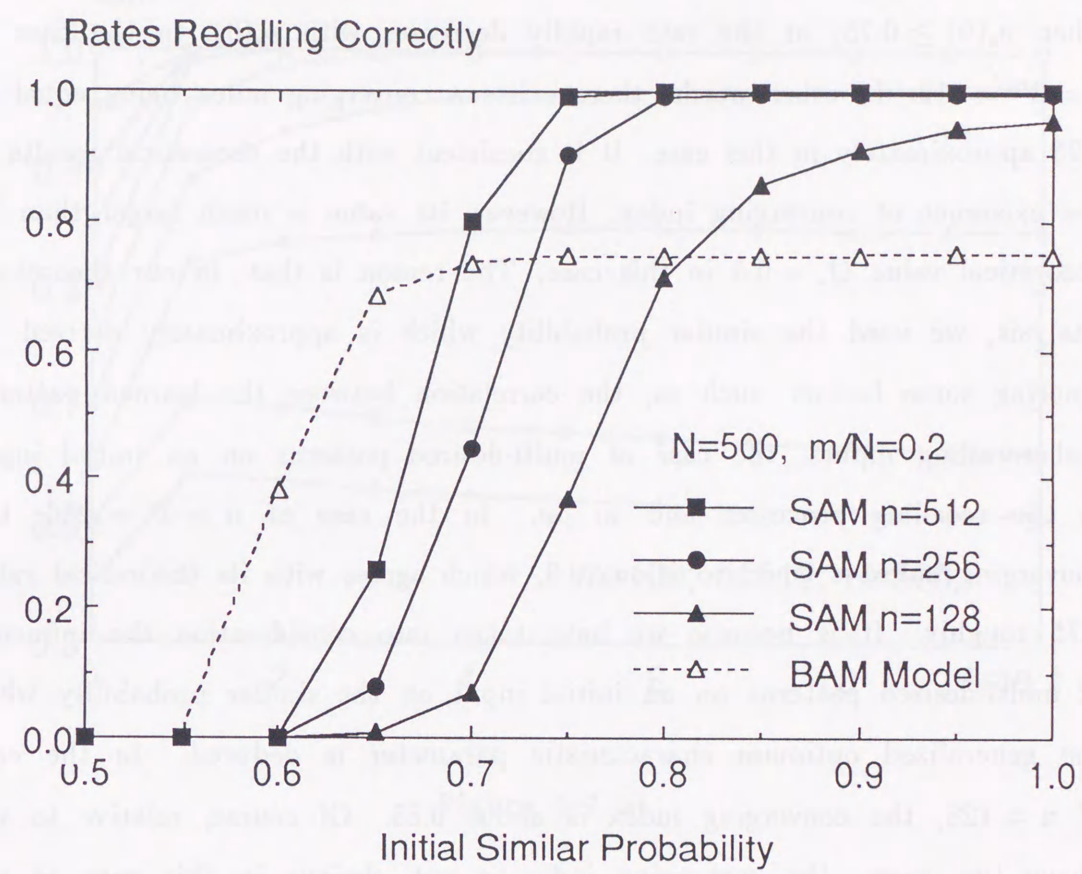



**Figure 5-6:** Successful rates in the recalling processes of SAM model and BAM model.

for $N = 500$, $m = 100$ and $n = 128$, 256 and 512. According to the theoretical analysis, the optimum characteristic parameter $n = 2^{\tau_1} = 512$ and the generalized optimum characteristic parameter $n = 2^{\tau} = 256$ in the case of $N = 500$. The experimental results are shown in Figure 5-6 and Figure 5-7.

We say a test is successful in our experiment if the desired pattern on the initial input of this test is retrieved correctly in the recalling processes or it is fail. Figure 5-6 shows the successful rates. From this figure, we can see that the rate is about 1 when $p_s(0) \geq 0.75$, viz., recalling outputs converge to the desired patterns on their initial inputs with probability one when $p_s(0) \geq 0.75$, or the rate rapidly decreases with $p_s(0)$, in the case of $n = 2^\eta = 512$. In other words, there exists a converging index being equal to 0.75 approximately in this case. It is consistent with the theoretical results of the existence of converging index. However, its value is much larger than its theoretical value $Q_\nu = 0.5$ in this case. The reason is that, in our theoretical analysis, we used the similar probability which is approximately derived by ignoring some factors, such as, the correlation between the learned patterns and recalling inputs, the case of multi-desired patterns on an initial input in the recalling processes and so on. In the case of $n = 2^r = 256$, the converging index is equal to about 0.8, which agrees with its theoretical value 0.75 roughly. It is because we have taken into consideration the influence of multi-desired patterns on an initial input on the similar probability while the generalized optimum characteristic parameter is deduced. In the case of $n = 128$, the converging index is about 0.85. Of course, relative to the above two cases, the converging index is not obvious in this case as the characteristic parameter is too small. We have also done the simulation experiments on BAM model for the numbers of neurons at neuron fields A and B being both equal to 500 and show the results in Figure 5-6 too. Like in Figure 5-4, we also assumed that, for any initial input, which neuron field contains the desired pattern on the initial input has been known before the recalling processes start to do.

Figure 5-7 shows the statistical value of the similar probability. Like in subsection 4.3.3, for examining the dependence of the similar probability on the initial similar probability $p_s(0)$, the initial inputs of all tests are divided into 11 groups by $q_a(0) = 0.5 + 0.05a$ $(a = 0, 1, \cdots)$. In each group, the mean $q_a(t)$ of $q^{(i)}(t)$, which is defined in equation (4.25), is calculated with equation (4.27). Figure 5.7 a, b and c show the means $q_a(t)$ in the case of $n = 512$, 256 and 128, respectively. As in Figure 5.6, from these figures, we can see



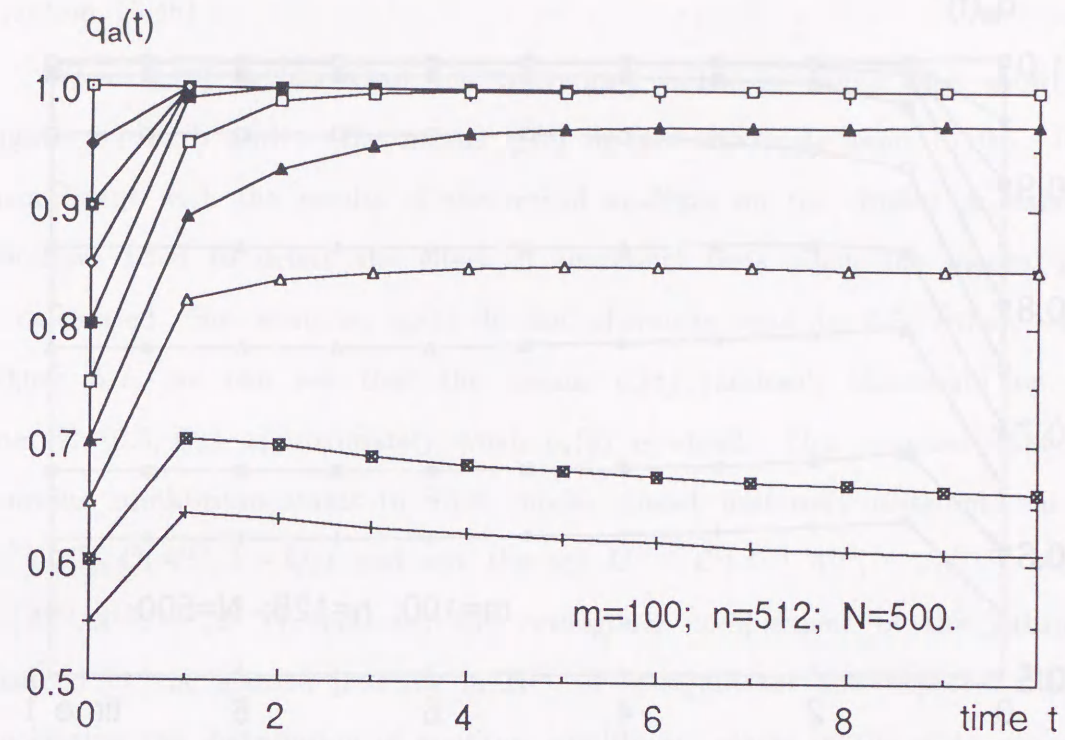



Figure 5-7 a:

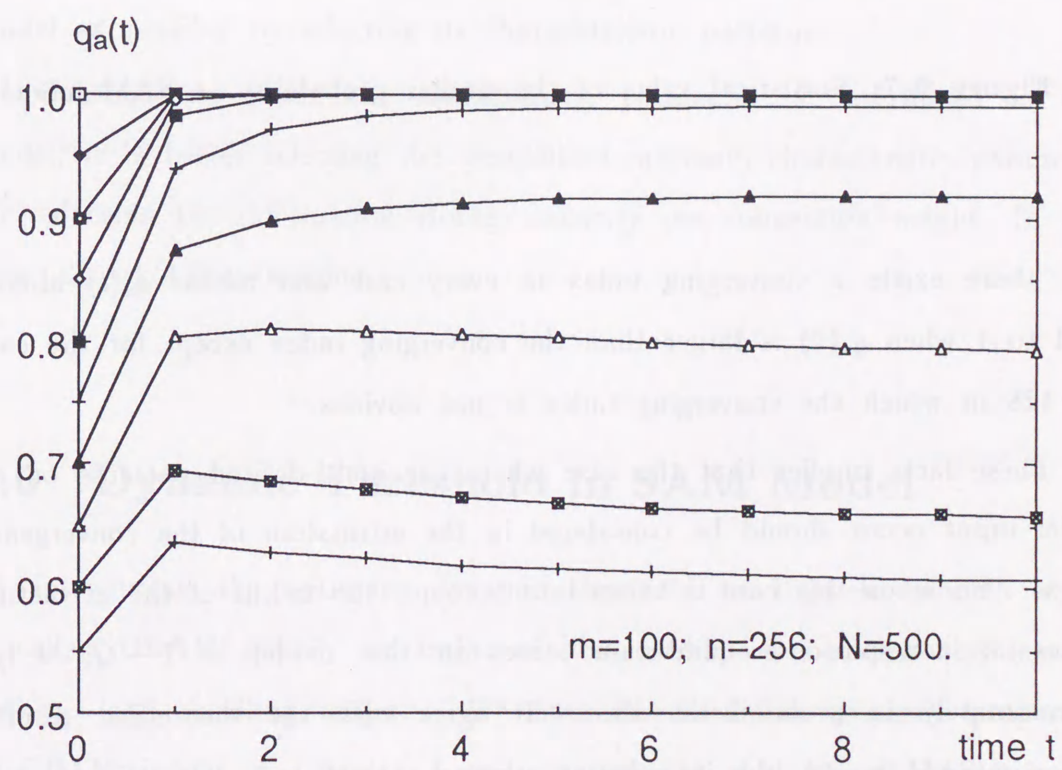



Figure 5-7 b:

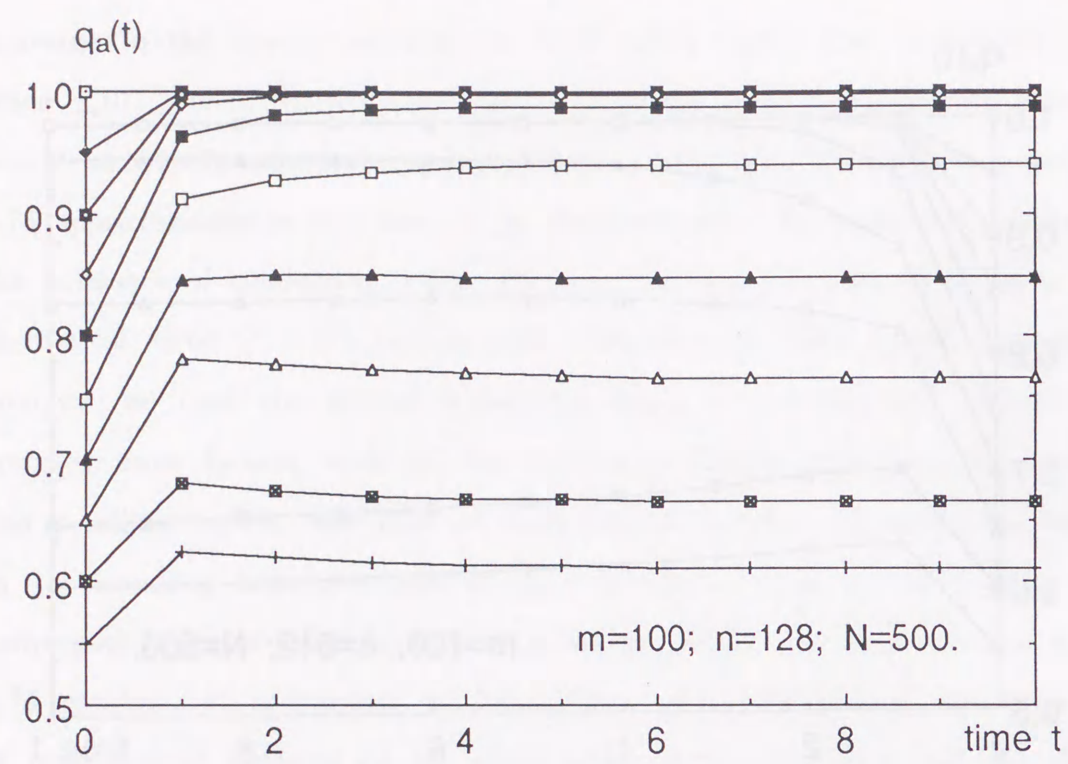



Figure 5-7 c:

**Figure 5-7:** Statistical value of the similar probability in SAM model.

that there exists a converging index in every case and means $q_a(t)$ almost tend to 1 when $q_a(0)$ is larger than the converging index except for the case $n = 128$ in which the converging index is not obvious.

These facts implies that the case where the multi-desired patterns on an initial input occur should be considered in the estimation of the convergence index. And when this case is taken into account, the radius of the attracting basins of non-spurious equilibrium states in this model is $1 - Q_\nu \approx 1/4$ approximately in probabilistic sense. It agree with the theoretical results. Thus, in SAM model, the information storage capacity per connection weight is not less than 1/4 when the characteristic parameter $n$ is decided by



equation (5.48).

When $q_a(0)$ is less than the converging index in every case shown in Figure 5-7 a, b and c, the means $q_a(t)$ do not obviously tend to 0.5. It is inconsistent with the results of theoretical analyses on the similar probability. We have tried to delete the effect of successful tests when the means $q_a(t)$ is calculated. But even so, $q_a(t)$ do not obviously tend to 0.5, either. From Figure 5-7, we can see that the means $q_a(t)$ randomly distribute on the interval $(0.5, Q_\nu)$ approximately when $q_a(0)$ is small. This suggests that the spurious equilibrium states in SAM model almost uniformly distribute on set $\mathrm{U}^N \setminus \cup_{\xi=1}^{m} O(A^{(\xi)}, 1 - Q_\nu)$ and not the set $\mathrm{U}^N \cap \mathcal{L}^c(A^{(1)}, A^{(2)}, \cdots, A^{(m)})$, where $\mathcal{L}^c(A^{(1)}, A^{(2)}, \cdots, A^{(m)})$ indicates the orthogonal complement of the subspace spanned by the learned patterns in $\mathbf{R}^N$. It is significant and expected clearly elucidating the distribution of spurious equilibrium states in the state space of SAM model and the dependence of the states on the characteristic patterns. It may be an effective way to reduce the spurious equilibrium states in SAM model as possible by selecting its charachteristic patterns.

Consequently, when the characteristic parameter is determined in this model, it is better selecting the generalized optimum characteristic parameter for enlarging the information storage capacity per connection weight. In this case, $S_c \geq 1/4$ (bits/weight).

## 5.6 Dynamic Threshold in SAM Model

By introducing the optimum dynamic threshold to Hopfield model, which is called ODAM model, we improved the correlation between the learned patterns and recalling outputs. We have discussed the dynamic properties of ODAM model in Chapter 4 and pointed out that the threshold in an associative memory model should be dynamic. SAM model is a kind of generalized associative memory model. In the above discussion about SAM model, we only considered that the threshold is zero. Here, like in Hopfield model, we shall discuss the optimum dynamic threshold in SAM model by means of the similar probability in this model.

Instead of the function given in equation (2.9), we define the function of the state evolutions of neurons in SAM model with dynamic threshold as follows.

$$\mathbf{X}(t+1) = f_\vartheta[\,,\mathbf{H}(t+1)] \qquad (t=0,1,\cdots) \tag{5.51}$$

where function $f_\vartheta(\cdot)$ is

$$f_\vartheta[h_j(t)] = \begin{cases} 1 & h_j(t) > i_j(t)n \\ x_j(t) & |h_j(t)| \le i_j(t)n \\ -1 & h_j(t) < -i_j(t)n \end{cases} \qquad (j=1,2,\cdots,N); \tag{5.52}$$

The vector $\mathbf{I}(t) = n\{i_1(t),\cdots,i_N(t)\}$ is the dynamic threshold in this model and $n$ is the characteristic parameter. In general, the threshold in every neuron can be different. Here, in order to guarantee that state evolutions of neurons in this model are independent of the order of neurons, we consider $\mathbf{I}(t) = i(t)n\{1,1,\cdots,1\}$ $(t=0,1,\cdots)$ and $i(t)$ to be a nonnegative undetermined function of time $t$ in the recalling processes. Next, making use of the statistical neurodynamics, we shall deduce the optimum function $i(t)$.

Like in Hopfield model, for arbitrary initial input $\mathbf{X}(0)$, assume that the desired pattern on $\mathbf{X}(0)$ is the learned pattern $A^{(\eta)}$, and then the similar probability in this model is

$$\begin{aligned} p(t+1) &= \mathbf{P}\left\{\mathbf{X}_j(t+1) = A_j^{(\eta)}\right\} \qquad (1 \le j \le N) \\ &= \mathbf{P}\left\{f_\vartheta[\theta_j(t) + \mathbf{Y}(t)O^{(\eta)T}A_j^{(\eta)}] = A_j^{(\eta)}\right\} \\ &= \mathbf{P}\left\{\theta_j(t) < \frac{2q(t)-1-i_j(t)}{2}n\right\} \end{aligned}$$



$$\begin{aligned} &+p(t)\mathbf{P}\left\{\frac{2q(t)-1-i_j(t)}{2}n \le \theta_j(t) \le \frac{2q(t)-1+i_j(t)}{2}n\right\} \\ \approx\ & \Phi\left(\frac{2q(t)-1-i(t)}{2\nu\sqrt{1-q(t)}}\right) \\ &+p(t)\left[\Phi\left(\frac{2q(t)-1+i(t)}{2\nu\sqrt{1-q(t)}}\right) - \Phi\left(\frac{2q(t)-1-i(t)}{2\nu\sqrt{1-q(t)}}\right)\right] \end{aligned} \tag{5.53}$$

where $q(t)$ is defined in equation (5.11). In accordance with the discussion in subsection 5.3.2, we have $q(t) \approx p(t)$ in the recalling processes. Hence, we can use $p(t)$ instead of $q(t)$ in above equation and obtain an approximate expression of similar probability as follows.

$$p(t+1) \approx (1-p(t))\Phi\left(\frac{2p(t)-1-i(t)}{2\nu\sqrt{1-p(t)}}\right) + p(t)\Phi\left(\frac{2p(t)-1+i(t)}{2\nu\sqrt{1-p(t)}}\right) \tag{5.54}$$

Clearly, when $i(t) = 0$, we have

$$p(t+1) = \Phi\left(\frac{2p(t)-1}{2\nu\sqrt{1-p(t)}}\right) \tag{5.55}$$

It is equation (5.17). Like in Chapter 4, using the value of function $p(t+1)$ and its derivative $[p(t+1)]'_i$ at $i(t)=0$, it can be proved that there exist $i(t) > 0$ such that the similar probability $p(t+1)$ is maximum for given $p(t)$. Further, solving equation

$$[p(t+1)]'_i = 0$$

we obtain that similar probability $p(t+1)$ is maximum approximately when the dynamic threshold

$$i(t)n = 2m\frac{1-p(t)}{p(t)} \tag{5.56}$$

This is the optimum dynamic threshold in the recalling processes of SAM model.

From equation (5.56), we can see

$$i(t)n \longrightarrow 0$$

as $p(t) \to 1$. Namely, when $p(t) \approx 1$, the optimum dynamic threshold $i(t)n$ is very small. On the other hand, in equation (5.54),

$$\frac{2p(t)-1}{2\nu\sqrt{1-p(t)}} \longrightarrow \infty$$

as $p(t) \to 1$. This suggests that the effect of the optimum dynamic threshold $i(t)n$ on the dynamic properties of SAM model is very limit.

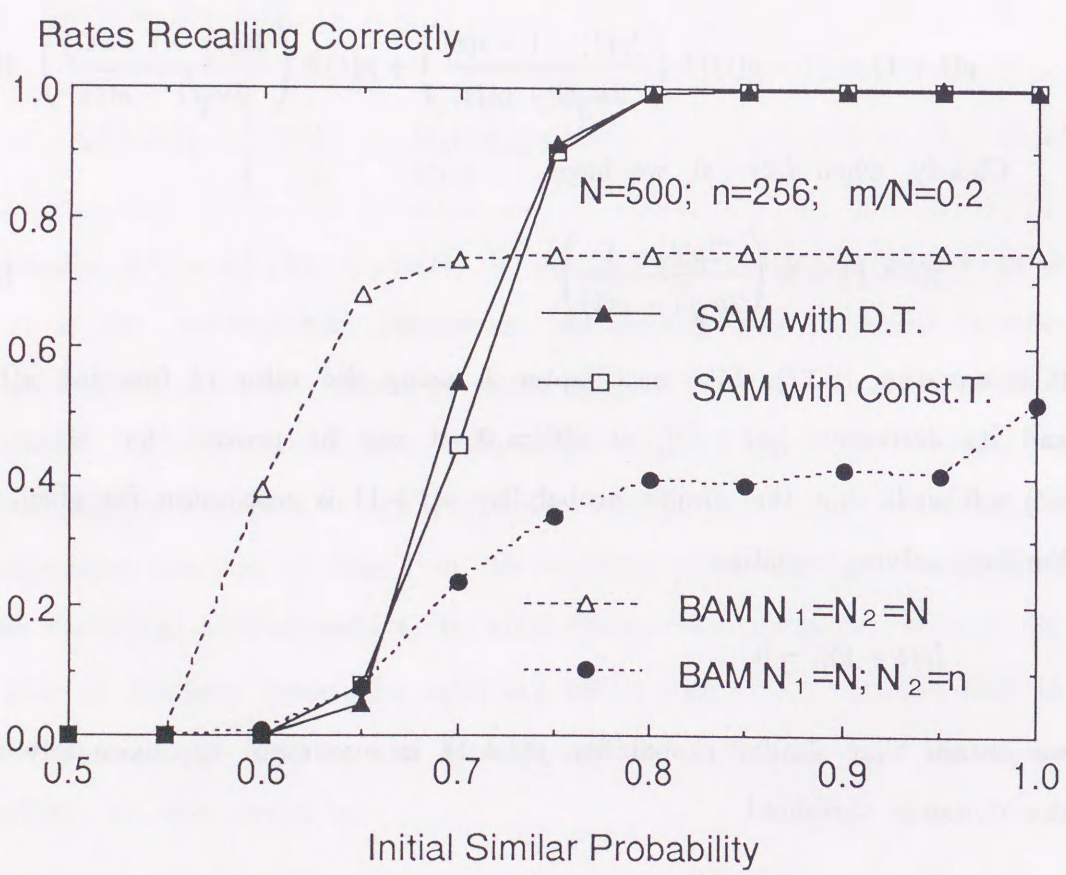



**Figure 5-8:** Successful rates in the recalling processes of the SAM model with the optimum dynamic threshold and zero threshold, and BAM model.



We have done a simulation experiment on the SAM model with the optimum dynamic threshold and the simulation results are shown in Figure 5-8. In our experiment, the numbers of neurons and learned patterns are $N = 500$ and $m = 100$, respectively. The characteristic parameter is $n = 256$. Figure 5-8 shows the successful rates in the recalling processes of the SAM models with the optimum dynamic threshold and the constant threshold zero, respectively. For convenience of comparison, the successful rates in the recalling processes of the BAM models with the neuron numbers $N = N_1 = N_2 = 500$ and $N_1 = N = 500$, $N_2 = n = 256$ are also shown in Figure 5-8. From this figure, we can see that the rates are hardly changed whether SAM model has the optimum dynamic threshold or zero threshold, except for the cases of $p(0) = 0.7$ and $0.75$. In the cases of $p(0) = 0.7$ and $0.75$, the rates have a little increase when SAM model has the optimum dynamic threshold. We also calculated the means of $q_a(t)$ defined in equation (4.27) in our experiments. Comparing it with the results shown in Figure 5-7 b, the means hardly changed. This confirmed that the effect of the dynamic threshold is limited in SAM model, which agrees with the theoretical result, too.

# Chapter 6

# Conclusions

In accordance with the principle objective stated in Chapter 1, this thesis has systematically discussed the discrete associative memory neural network with feedback by means of statistical neurodynamics and Lyapunov function.

Firstly, in Chapter 2, we extended the concept of the global stability of associative memory model to the periodical global stability. By the constructing examples, we investigated the global stability of Hopfield model, elucidated this model is not unconditionally globally stable and the largest period of the periodical global stability of Hopfield model with $N$ neurons can be as large as $2N$ at least if the connection weights matrix is not symmetrical. We also pointed out the ambiguity of the recalling outputs of BAM model, which depends on the direction that the recalling start to do. Following these discussions, an generalized associative memory model was proposed. The generalized associative memory model is unconditionally globally stable and has not the problems still remaining in the conventional associative memory models. In a sense, this model can be considered as a combination of Hopfield model and BAM model. For analyzing the fundamental properties of associative memory model by statistical neurodynamics, a statistical variable—similar probability in the recalling processes of associative memory model was defined and some fundamental results in terms of this probability were given in Chapter 3. We estimated the similar probability in Hopfield model approximately and demonstrated that the memory capacity of Hopfield model with $N$ neurons is not larger than $2N/\pi$ by means of this probability.

In order to reduce the dependence of the recalling outputs on all learned patterns and improve the strange shape of equilibrium state attracting basins in Hopfield model, Chapter 4 introduced a dynamic threshold into this model. By means of statistical neurodynamics, we deduced the optimum dynamic threshold and succeeded in proving that the Hopfield model with the optimum dynamic threshold, called ODAM model, is of better dynamic properties and larger memory capacity. Simulations on the ODAM model have been done and the simulation results confirmed our theoretical analysis results.

Chapter 5 proposed a semi-orthogonal training algorithm for constructing the connection weight matrix in the generalized associative memory model. Though this training algorithm is based on the Hebbian learning rule, the partial from the correlation among the learned patterns in the error terms of recalling processes can be greatly reduced by this training algorithm. The generalized associative memory model with the connection weight matrix constructed by this training algorithm is referred to as SAM model. Like in Hopfield model, a Lyapunov function was successfully used as a tool to prove SAM model is unconditionally globally stable. We analyzed the convergent properties of this model and gave a convergent criteria—converging index of SAM model. Making use of the converging index, we derived the optimum characteristic parameter regarding SAM model. The memory capacity of SAM model with $N$ neurons is:

$$m_c = \frac{N}{2 \ln \ln N}$$

The information storage capacity per connection weight $S_c$ satisfies the following inequality:

$$\frac{1}{2} \geq S_c \geq \frac{1}{4} \quad \text{(bits/weight)}$$



when the characteristic parameter in this model is decided by equation (5.48). Furthermore, the dynamic threshold is discussed as well and its limitation in SAM model is explained.

In the theoretical analysis on the dynamic properties of associative memory model, we mainly used the statistical neurodynamics. Generally, the dynamic behavior of neural network is not fixed or accurate. It is of fuzziness and subject to some statistical rules. Therefore, using the statistical neurodynamics to analyze the dynamic properties of associative memory model is reasonable and significant.

However, from the discussions in Chapter 3, 4 and 5, we have seen that using the similar probability

$$p(t) = P\{x_j(t) = A_j^{(\eta)}\},$$

which are approximately estimated, often led to a disagreement between the theoretical analysis results and simulation results when the initial similar probability is small and the parameter $\lambda = \sqrt{m/N}$ is large. For example, theoretical analyses show that the similar probability is independent of the initial similar probability, but experiment results show it closely depend on the initial similar probability. This disagreement all occurred in Hopfield model, ODAM model and SAM model although it is not clear in SAM model when $\lambda = \sqrt{m/N}$ and $\nu = \sqrt{m/n}$ are small. S.Amari et al[14] made a comparatively accurate estimation on the similar probability in the recalling processes of Hopfield model by considering the dependence of the recalling outputs on learned patterns. They succeeded in confirming that the similar probability in the recalling processes of Hopfield model closely depends on the initial similar probability, and obtained a consistent results between theoretical analyses and simulation experiments when $\lambda = \sqrt{0.08}$ and $\sqrt{0.2}$, respectively. But H.Nishimori[151] pointed out the theoretical results given by S.Amari et al is only reasonable in the case that $\lambda$ is small.

We also have tried to estimated the similar probability in ODAM model and SAM model by considering the dependence of the recalling outputs on learned patterns. Even it is done, the theoretical value of similar probabilities in ODAM model and SAM model are almost unchanging. This conversely explains that only considering the dependence is insufficient when the similar probability is estimated. Further, the limitation of the dynamic threshold in SAM model also suggests that only considering the dependence is insufficient when the similar probability is calculated, because dynamic threshold can efficiently reduce the dependence.

Rigorously, it is not true to give the probability distribution of the error term in the recalling processes by using De Moivre-Laplace limit theorem, because the error term is not a sum of independent random variables. Fortunately, when $\lambda$ and $\nu$ are small, the dependence between these random variables are very weak and can be approximately neglected. This fact have been confirmed by the approximately consistent results between theoretical analysis and simulation experiment.

In general, when the similar probability in an associative memory model is estimated, the following factors should be considered as well .

- The influence of the case of multi-desired patterns on an initial input on the similar probability.
- A recalling output at time $t+1$ is subject to the statistical rules or not depends on whether the recalling output at time $t$ has converged to a equilibrium state. Similar probability will not change with time again when a recalling output has converged.

These two factors imply that the similar probability depends on the distribution of equilibrium states in associative memory model. Considering the two factors in the estimation of similar probability is an problem expected to solve.

In Chapter 2, we have pointed out that the generalized associative memory model is a neural network with a hidden layer for extracting characteristics. We can also consider to extend the one layer to multi layers for sufficiently extracting characteristics and structure a multi-layers associative memory model with feedback. It may be an efficient way to construct large and high-effective neural networks.



To use neural network as a tool to solve the problems, such as, pattern recognition and classification, it is necessary discriminating what the recalling outputs converge to. In other words, we must give a discriminating condition in an associative memory model for judging whether the model equilibrate to the desired patterns on their initial inputs. We have tried to discuss this problem and proposed an equal-potential condition in the recalling processes of SAM model. We have explicated this condition is not necessary and sufficient rigidly, but, except for a few special examples, it is necessary and sufficient almost. Computer simulations show that the successful judging rates can be larger than 99.8% if the equal-potential is moderately determined. But there are some detail theoretical problems expected to solve. Furthermore,

- Expanding the generalized associative memory model to continuous form.
- Looking for the optimum characteristic extracting mapping for given task, in particular, multi-layers case.
- Optimizing the semi-orthogonal training algorithm to improve the dynamic behavior of the model. For example, X.Zhuang[206] et al discussed how to make good the training algorithm in BAM model and their simulation results confirmed that the memory capacity of BAM model with an improved connection weight matrix is about $N$. This is close to the upper bound given by Y.Abu[1]. How are these algorithms introduced into the generalized associative memory model?

all of these are also significant theoretical and practical problems for generalized associative memory model and expected to solve.

# Bibliography

[1] Y. S. Abu-Mostafa and J. Jacques. "Information Capacity of the Hopfield Model". *Scientific American*, pages 88–95, March 1985.

[2] Y. S. Abu-Mostafa and D. Psaltis. "Optical Neural Computers". *IEEE Trans. Inform. Theory*, IT–31:461–464, April 1987.

[3] D. H. Ackley and G. E. Hinton. "A Learning Algorithm for Boltzmann machines". *Cognitive Science*, vol. 9:147–169, 1985.

[4] S. B. Aiyer, M. Niranjan, and F. Fallside. "A Theoretical Investigation into the Performance of the Hopfield Model". *IEEE Trans. Neural Networks*, NN–1(2):204–215, June 1990.

[5] K. Akazawa and K. Kato. "Neural Network Model for Control of Muscle Force Based on the Size Principle of Motor Unit". *Proceedings of the IEEE*, vol. 78(9):1531–1535, September 1990.

[6] R. B. Allen and J. Alspector. "Learning of Stable in Stochastic Asymmetric Networks". *IEEE Trans. Neural Networks*, NN–1(2):216–222, June 1990.

[7] L. B. Almeida. "A Learning Rule for Asynchronous Perceptrons with Feedback in a Combinatorial Environment". In *Proceedings of the 1st IEEE International Conference on Neural Networks (ICNN-87)*, volume II, pages 609–618, 1987.

[8] S. Amari. "Characteristics of Randomly Connected Threshold Element Networks and Network Systems". In *Proc. IEEE, 59*, pages 35–47, 1971.

[9] S. Amari. "Characteristics of Randomly Nets of Analog Neuron-like Elements". *IEEE Trans. Syst., Man, Cybern.*, SMC–2:643–657, January 1972.

[10] S. Amari. "Neural Theory of Association and Concept Formation". *Biological Cybernetics*, vol. 26:175–185, 1977.

[11] S. Amari. *The Mathematical Physics in Neural Networks.* SANGYO TOSHO, 1987. (in Japanese).

[12] S. Amari. "Mathematical Foundations of Neurocomputing". *Proceedings of the IEEE*, vol. 78(9):1443–1463, September 1990.

[13] S. Amari, K. Kurata, and H. Nagaoka. "Information Geometry of Boltzmann Machines". *IEEE Trans. Neural Networks*, 3(2):260–271, March 1992.

[14] S. Amari and K. Maginu. "Statistical Neurodynamics of Associative memory". *Neural Networks*, 1:63–73, January 1988.

[15] D. J. Amit, H. Gutfreund, and H. Sompolinsky. "Spin-glass models of neural networks". *Physics Review A*, vol. 2(10):1007–1018, May 1985.

[16] D. J. Amit, H. Gutfreund, and H. Sompolinsky. "Storing infinite numbers of patterns in a spin-glass model of neural networks". *Physics Review Letters*, 55:1530–1533, August 1985.

[17] B. D. O. Anderson and J. B. Moore. *Optimal Filtering.* Prentice Hall, Englewood Cliffs, NJ, 1979.

[18] J. A. Anderson. "A Simple Neural Network Generating an Associative Memory". *Mathematical Biosciences*, vol. 14:197–220, 1972.



[19] J. A. Anderson. "Cognitive and Psychological Computations with Neural Models". *IEEE Trans. Syst., Man, Cybern.*, SMC–13:799–815, September 1983.

[20] J. A. Anderson and E. Rosenfeld. *Neurocomputing: Foundations of Research.* MIT Press, Cambridge MA, 1988.

[21] J. A. Anderson, J. W. Silverstein, S. R. Ritz, and R. S. Jones. "Distinctive Features, Categorical Perception, and Probability Learning: Some Applications of a Neural Model". *Psychological Review*, 84:413–451, 1977.

[22] M. A. Arbib. *The Metaphorical Brain 2: Neural Networks and Beyond.* John Wiley & Sons, 1989.

[23] R. A. Athale. "Optical implementation of associative memory with controlled nonlinearity in the correlation domain". *Optics Letters*, vol. 11(7):482–484, July 1986.

[24] L. E. Atlas and Y. Suzuki. "Digital Systems for Artificial Neural Networks". *IEEE Circuits and Devices Magazine*, pages 20–24, November 1989.

[25] E. Barnard and D. Casasent. "Invariance and Neural Nets". *IEEE Trans. Neural Networks*, NN–2:498–508, September 1991.

[26] P. L. Bartlett and T. Downs. "Using Random Weights to Train Multilayer Networks of Hard-Limiting Units". *IEEE Trans. Neural Networks*, NN–3:202–210, March 1992.

[27] A. G. Barto, R. S. Sutton, and C. W. Anderson. "Neuronlike Elements That Can Solve Difficult Control Problems". *IEEE Trans. Syst., Man, Cybern.*, SMC–13:834–846, September 1983.

[28] M. L. Brady, R. Raghavan, and J. Slawny. "Back Propagation Fails to Separate Where Perceptrons Succeed". *IEEE Transactions on Circuits and Systems*, CAS–36(5):665–674, May 1989.

[29] R. Braham and J. O. Hamblen. "The Design of a Neural Network with a Biologically Motivated Architecture". *IEEE Trans. Neural Networks*, NN–1:251–262, September 1990.

[30] J. Bruck. "On the Convergence Properties of the Hopfield Model". *Proceedings of the IEEE*, 78(10):1579–1585, October 1990.

[31] J. Bruck and M. Blaum. "Neural Networks, Error-Correcting Codes, and Polynomials over the Binary n-Cube". *IEEE Trans. Inform. Theory*, IT–35(5):976–987, September 1989.

[32] D. J. Burr. "Experiments with a Connectionist Text Reader". In *Proceedings of the 1st IEEE International Conference on Neural Networks (ICNN-87)*, volume 4, pages 717–724, 1987.

[33] E. R. Caianiello. "Outline of a theory of thought-processes and thinking machines". *Journal of Theory Biology*, 2:204–235, 1961.

[34] G. A. Carpenter, M. A. Cohen, and S. Grossberg. "Computing with Neural Networks". *Science*, 235:1226–1227, March 1987.

[35] G. A. Carpenter and S. Grossberg. "A Massively Parallel Architecture for a Self–Organizing Neural Pattern Recognition Machine". *Computer Vision, Graphics, and Image Processing*, 37:54–115, 1987.

[36] G. A. Carpenter and S. Grossberg. "ART 2: Self–Organization of Stable Category Recognition Codes for Analog Input Patterns". *Applied Optics*, 26:4919–4930, December 1987.



[37] F.-C. Chen. "Back-Propagation Neural Networks for Nonlinear Self-Tuning Adaptive Control". *IEEE Control Systems Magazine*, pages 44–48, April 1990.

[38] K. L. Chung. *A Course in Probability Theory*. Academic Press, New York, 1974.

[39] M. A. Cohen and S. Grossberg. "Absolute Stability of Global Pattern Formation and Parallel Memory Storage by Competitive Neural Networks". *IEEE Trans. Syst., Man, Cybern.*, SMC–13:815–826, September 1983.

[40] M. A. Cohen and S. Grossberg. "Masking Fields: A Massing Parallel Neural Architecture for Learning, Recognizing, and Predicting Multiple Groupings of Patterned Data". *Applied Optics*, 26:1866, 1987.

[41] L. N. Cooper, F. Liberman, and E. Oja. "A Theory for the Acquisition and Loss of Neuron Specificity in Visual Cortex". *Biological Cybernetics*, 33:9–28, 1979.

[42] S. Dasgupta. "Convergence in Neural Memories". In *Proc.1987 IEEE International Symp. On Circuits and Systems IEEE.*, pages 366–369, New York. USA, 1987.

[43] R. E. Dayhoff and J. E. Dayhoff. "Neural Networks for Medical Image Processing". In *Proceedings of IEEE SCAMC.*, pages 271–275, 1988.

[44] A. Dembo and T. Kailath. "Model-Free Distributed Learning". *IEEE Trans. Neural Networks*, NN–1(1):58–70, March 1990.

[45] E. Domany, J. L. van Hemmen, and K. Schulten (eds.). *Models of Neural Networks*. Springer-Verlag New York, 1991.

[46] O. K. Ersoy and C.-H. Chen. "Learning of Fast Transforms and Spectral Domain Neural Computing". *IEEE Transactions on Circuits and Systems*, CAS-36(5):704–712, May 1989.

[47] G. Fant. *Speech Sounds and Features*. MIT Press, 1973.

[48] N. H. Farhat, D. Psaltis, A. Prata, and E. Paek. "Optical implementation of the Hopfield Model". *Applied Optics*, vol. 24:1469–1475, 1985.

[49] N. H. Farhat, D. Psaltis, A. Prata, and E. Paek. "Optoelectronic Neural Networks and Learning Machines". *IEEE Circuits and Devices Magazine*, pages 32–41, September 1989.

[50] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1–2. JOHN WILLEY, 1970.

[51] M. Fukumi, S. Omatsu, F. Takeda, and T. Kosaka. "Rotation-Invariant Neural Pattern Recognition System with Application to Coin Recognition". *IEEE Trans. Neural Networks*, 3(2):272–279, March 1992.

[52] K. Fukushima. "Cognitron: A Self-Organizing Multilayered Neural Network". *Biological Cybernetics*, vol. 20:121–136, 1975.

[53] K. Fukushima. "Neocognitron: A Self-Organizing Multilayered Neural Network Model for a Mechanism of Pattern Recognition Unaffected by Shift in Position". *Biological Cybernetics*, vol. 36:193–202, 1980.

[54] K. Fukushima. "A Hierarchical Neural Network Model for Associative Memory". *Biological Cybernetics*, vol. 50:105–113, 1984.

[55] K. Fukushima. "A Neural Network Model for Selective Attention in Visual Pattern Recognition". *Biological Cybernetics*, vol. 55:5–15, 1986.



[56] K. Fukushima. "Neural-Network Model for Selective Attention in Visual Pattern Recognition and Associative Recall". *Applied Optics*, 26:4985–4992, December 1987.

[57] K. Fukushima, S. Miyake, and T. Ito. "Neocognitron: A Neural Network Model for a Mechanism of Visual Pattern Recognition ". *IEEE Trans. Syst., Man, Cybern.*, SMC–13:826–834, September 1983.

[58] K. Funabashi. "On the Approximate Realization of Continuous Mapping by Neural Networks". *Technical report of IEICE*, MBE–9, 1988.

[59] K. Funabashi. "On the Approximate Realization of Identity Mappings by Three-Layer Neural Networks". *Technical report of IEICE*, MBE–174, 1988.

[60] S. I. Gallant. "Perceptron-Based Learning Algorithms". *IEEE Trans. Neural Networks*, NN–1(2):179–191, June 1990.

[61] E. Gloes-Chace, F. Fogelman-Soulie, and D. Pellegrin. "Decreasing Energy Functions as a Tool for Studying Threshold Network". *Discrete Mathematics*, vol. 12:261–277, 1985.

[62] M. A. Gluck and D. B. Parker E. S. Reinfsnider. "Some Biological Implications of a Differential-Hebbian Learning Rule". *Psychobiology*, 16:298–302, 1988.

[63] D. L. Gray and A. N. Michel. "A Training Algorithm for Binary Feedforward Neural Networks". *IEEE Trans. Neural Networks*, NN–3(2):176–194, March 1992.

[64] G. R. Grimmett and D. R. Stirzaker. *Probability and Random Processes*. Oxford Science Publications, 1992.

[65] S. Grossberg. "Adaptive Pattern Classification and Universal Recoding: I. Parallel Development and Coding of Neural Feature Detectors". *Biological Cybernetics*, 23:187–202, 1977.

[66] S. Grossberg. *Studies of Mind and Brain*. Reidel, Boston, 1982.

[67] S. Grossberg. "Competitive Learning: From Interactive Activation to Adaptive Resonance". *Cognitive Science*, 11:23–63, 1987.

[68] S. Grossberg. "Nonlinear Neural Networks: Principles, Mechanisms, and Architectures". *Neural Networks*, pages 17–61, 1988.

[69] S. Grossberg and E. Mingolla. "Neural Dynamics of Form Perception: Boundary Completion, Illusory Figures, and Neon Color Spreading". *Psychological Review*, 92:173–211, 1985.

[70] S. Grossberg and E. Mingolla. "Neural Dynamics of Perceptual Grouping: Textures, Boundaries, and Emergent Segmentation". *Perception and Psychophysics*, 38:141–171, 1985.

[71] C. C. Guest and R. Tekolste. "Designs and Devices for Optical Bidirectional Associative Memories". *Applied Optics*, 26:5055–5060, December 1987.

[72] K. Haines and R. Hecht-Nielsen. "A BAM with Increased Information Storage Capacity ". In *Proceedings of the IEEE International Conference on Neural Networks (ICNN-88)*, volume 1, pages 181–190, 1988.

[73] J. B. Hampshire and A. H. Waibel. "A Novel Objective Function for Improved Phoneme Recognition Using Time-Delay Neural Networks". *IEEE Trans. Neural Networks*, NN–1(2):216–228, June 1990.

[74] E. M. Harth, T. J. Csermely, B. Beek, and R. D. Lindsay. "Brain functions and neural dynamics". *Journal of Theoretical Biology*, 26:93–120, January 1970.



[75] M. H. Hassoun and A. M. Youssef. "High performance recording algorithm for Hopfield model associative memories". *Optical Engineering*, vol. 28(1):46–54, January 1989.

[76] S. Haykin and C. Deng. "Classification of Radar Clutter Using Neural Networks". *IEEE Trans. Neural Networks*, NN–2(6):589–600, November 1991.

[77] D. O. Hebb. *The Organization of Behavior*. Wiley, New York, 1949.

[78] R. Hecht-Nielsen. *Neurocomputing*. Addison-Wesley, New York, 1989.

[79] T. H. Hildebrandt. "Optical Training of Threshold Linear Correlation Classifiers". *IEEE Trans. Neural Networks*, NN–2(6):577–588, November 1991.

[80] L. Holmstrom and P. Koistinen. "Using Additive Noise in Back-Propagation Training". *IEEE Trans. Neural Networks*, NN–3:24–38, January 1992.

[81] J. J. Hopfield. "Neural Networks and Physical Systems with Emergent collective computational abilities". In *Proceedings of the National Academy of Science. USA*, volume 79, pages 2554–2558, 1982.

[82] J. J. Hopfield. "Neuron with graded response have collective computational properties like those of two-states neurons". In *Proceedings of the National Academy of Science. USA*, volume 81, pages 3088–3092, 1984.

[83] J. J. Hopfield, D. I. Feinstein, and R. G. Palmer. "Unlearning Has a Stabilizing Effect in Collective Memories". *Nature*, 304:158–159, 1988.

[84] J. J. Hopfield and D. W. Tank. " 'Neural' Computation of Decisions in Optimization Problems". *Biological Cybernetics*, 52:141–152, 1985.

[85] J. J. Hopfield and D. W. Tank. "Computing with Neural Circuits: A Model ". *Science*, 233:625–633, 1986.

[86] D. Horn and M. Usher. "Neural Networks with Dynamical Thresholds ". *Physics Review A*, vol. 40:1036, 1986.

[87] S.-C. Huang and Y.-F. Huang. "Learning Algorithms for Perceptrons Using Back-Propagation with Selective Updates". *IEEE Control Systems Magazine*, pages 56–61, April 1990.

[88] X.-M. Huang and Y. Miyazaki. "Capacity of Semi-Orthogonally Associative Memory Model". *(Submitted to: IEICE Trans. Inf. Syst.)*.

[89] X.-M. Huang and Y. Miyazaki. "Generalization of Connected Neural Networks and Semi-Orthogonally Learning Method". In *The Third Researching Meeting on Ultra high-speed, and Ultra parallel Optic-Electronics Device of MONBUSYO Important Research Project*, pages 75–78, December 1991.

[90] X.-M. Huang and Y. Miyazaki. "Connected Associative Memory Neural Network with Dynamical Threshold Function". *IEICE Trans. Inf. Syst.*, E75–D(1):170–179, January 1992.

[91] X.-M. Huang and Y. Miyazaki. "The Capacity of the Semi-Orthogonally Associative Memories Model". In *Proc. Int. Joint Conference on Neural Networks (IJCNN-93), Nagoya,*, volume III, pages 2666–2669, October 1993.

[92] X.-M. Huang and Y. Miyazaki. "Semi-Orthogonally Associative Memory Neural Network Model". *IEICE Trans. Inf. Syst.*, J77–D–II(1):228–235, January 1994.



[93] Y. Ichikawa and T. Awa. "Neural Network Application for Direct Feedback Controllers". *IEEE Trans. Neural Networks*, NN–3(2):224–231, March 1992.

[94] M. Ishikawa, N. Mukohzaka, H. Toyoda, and Y. Suzuki. "Optical associatron: a simple model for optical associative memory". *Applied Optics*, vol. 28:291–301, January 1989.

[95] J.-S. Jang, S.-Y. Shin, and S.-Y. Lee. "Optical neural-net analog-to-digital converter". *Optics Letters*, vol. 14(3):159–161, February 1989.

[96] T. L. Jong and H. M. Tai. "Hopfield Model with Complementary Binary Representations". *Electronics Letters*, 24:1224–1225, September 1988.

[97] R. E. Kalman. "A New Approach to Linear Filtering and Prediction Problems". *Transactions of the ASME. Journal of Basic Engineering Series D*, 82:35–46, 1960.

[98] R. E. Kalman and R. Bucy. "New Results in Linear Filtering and Prediction Problems". *Transactions of the ASME. Journal of Basic Engineering Series D*, 83:95–108, 1961.

[99] M. Kam, R. Cheng, and A. Guez. "Pattern Retrieval and Learning in Nets of Asynchronous Binary Threshold Elements". *IEEE Transactions on Circuits and Systems*, CAS–36(3):353–364, 1989.

[100] J. A. Kangas, T. Kohonen, and J. T. Laaksonen. "Variants of Self-Organizing Maps". *IEEE Trans. Neural Networks*, NN–1:93–99, January 1990.

[101] J. M. Kinser, H. J. Caulfield, and J. Shamir. "Design for a Massive All-Optical Bidirectional Associative Memory: The Big BAM". *Applied Optics*, 27:3442–3443, August 1988.

[102] A. H. Klopf. "Drive-Reinforcement Learning: A Real-Time Learning Mechanism for Unsupervised Learning". In *Proceedings of the IEEE International Conference on Neural Networks (ICNN-88)*, volume 1, pages 441–445, 1988.

[103] T. Kohonen. *Content Addressable Memories*. Springer Verlog, 1987.

[104] T. Kohonen. "An Introduction to Neural Computing". *Neural Networks*, 1:3–16, January 1988.

[105] T. Kohonen. *Self Organization and Associative Memory*. Springer Verlog, New York, 2nd ed. edition, 1988.

[106] T. Kohonen. "The Self-Organizing Map". *Proceedings of the IEEE*, vol. 78(9):1464–1480, September 1990.

[107] S.-G. Kong and B. Kosko. "Differential Competitive Learning for Centered Estimation and Phoneme Recognition". *IEEE Trans. Neural Networks*, 2:118–124, January 1991.

[108] S.-G. Kong and B. Kosko. "Differential Competitive Learning for Centroid Estimation and Phoneme Recognition". *IEEE Trans. Neural Networks*, 3(2):211–223, March 1992.

[109] B. Kosko. "Differential Hebbian Learning". In *Proceedings American Institute of Physics: Neural Networks for Computing*, pages 277–282, 1986.

[110] B. Kosko. "Adaptive Bidirectional Associative Memories". *Applied Optics*, 26:4947–4960, December 1987.

[111] B. Kosko. "Bidirectional Associative Memories". *IEEE Trans. Syst., Man, Cybern.*, SMC–18(1):49–60, January 1988.



[112] B. Kosko. "Feedback Stability and Unsupervised Learning". In *Proceedings of the IEEE International Conference on Neural Networks (ICNN-88)*, volume 1, pages 141–152, 1988.

[113] B. Kosko. "Unsupervised Learning in Noise". *IEEE Trans. Neural Networks*, NN–1(1):44–57, March 1990.

[114] B. Kosko. "Stochastic Competitive Learning". *IEEE Trans. Neural Networks*, NN–2(5):522–529, September 1991.

[115] B. Kosko. *Neural Networks for Signal Processing*. Prentice Hall, Englewood Cliffs, NJ, 1991.

[116] B. Kosko. *Neural Networks and Fuzzy Systems*. Prentice Hall, Englewood Cliffs, NJ, 1992.

[117] A. V. Krishnamoorthy, G. Yayla, and S. C. Esener. "A Scalable Optoelectronic Neural System Using Free-Space Optical Interconnects". *IEEE Trans. Neural Networks*, NN–3(3):404–413, May 1992.

[118] C.-M. Kuan and K. Hornik. "Convergence of Learning Algorithms with Constant Learning Rates". *IEEE Trans. Neural Networks*, NN–2(5):484–489, September 1991.

[119] S. Kurogi. "Speech Recognition by An Artificial Neural Network Using Findings on the Different Auditory System". *Biol. Cybern.*, vol. 64:243–249, 1991.

[120] P. Lalanne, P. Chavel, and J. Taboury. "Optical inner-product implementation of neural network models". *Applied Optics*, vol. 28(2):377–385, January 1989.

[121] P. Lalanne, J. Taboury, and P. Chavel. "A proposed generalization of Hopfield's Algorithm". *Optics Communications*, vol. 63(1):21–25, July 1987.

[122] M. Lemmon and B. V. K. Vijaya Kumar. "Competitive Learning with Generalized Winner-Take-All Activation". *IEEE Trans. Neural Networks*, NN–3(2):167–175, March 1992.

[123] J-H Li, A.N.Michel, and W.Porod. "Analysis and synthesis of a class of neural networks: variable structure systems with infinite gain". *IEEE Transactions on Circuits and Systems*, CAS–36(5):713–731, May 1989.

[124] R. P. Lippman. "A Introduction to Computing with Neural Nets". *IEEE Transactions on Acoustics, Speech and Signal Processing*, pages 4–21, April 1987.

[125] R. P. Lippman and B. Gold. "Neural Classifiers Useful for Speech Recognition". In *Proceedings of the 1st IEEE International Conference on Neural Networks (ICNN-87)*, volume 4, pages 417–425, 1987.

[126] S. P. Luttrell. "Derivation of a Class of Training Algorithms". *IEEE Trans. Neural Networks*, NN–1(2):229–232, June 1990.

[127] J. K. Mangis, R. B. Voas, W. T. Zink, and K. Tiplitz Blackwelland T. P. Vogl. "Performance Comparison of a Neural Network with Human Observers on a Visual Target Detection Task". *Biological Cybernetics*, vol. 62:185–191, 1990.

[128] R. J. Marks, S. Oh, and L. E. Atlas. "Alternating Projection Neural Networks". *IEEE Transactions on Circuits and Systems*, CAS–36(6):846–857, June 1989.

[129] T. M. Martinetz, H. J. Ritter, and K. J. Schulten. "3D Neural Net for Learning Visuomotor Coordination of a Robot Arm". In *Proceedings of the International Joint Conference on Neural Networks (IJCNN-89)*, volume II, pages 351–356, 1989.



[130] G. Mathai and B. R. Upadhaya. "Performance Analysis and Application of the Bidirectional Associative Memory to Industrial Spectral Signatures". In *Proceedings of the International Joint Conference on Neural Networks (IJCNN-89)*, volume I, pages 33–37, 1989.

[131] K. Matsuoka. "An Associative Network with Cross Inhibitory Connections". *Biol. Cybern.*, vol. 61:393–399, 1989.

[132] A. D. McAulay. *Optical Computer Architectures.* John Wiley & Sons, 1991.

[133] W. S. McCulloch and W. A. Pitts. "A logical calculus of the ideas imminent in nervous activity". *Bulletin of Mathematical Biophysics*, 5:115–133, 1943.

[134] R. J. Mceliece, E. C. Posner, E. R. Rodemich, and S. S. Venkatesh. "The Capacity of the Hopfield Associative Memory". *IEEE Trans. Inform. Theory*, IT–33:461–482, July 1987.

[135] K. G. Mehrotra, C. K. Mohan, and S. Ranka. "Bounds on the Number of Samples Needed for Neural Learning". *IEEE Trans. Neural Networks*, NN–2(6):548–558, November 1991.

[136] J. M. Mendel. *Lessons in Digital Estimation Theory.* Prentice Hall, Englewood Cliffs, NJ, 1987.

[137] A. Michel and J. A. Farrell. "Associative Memories via Artificial Neural Networks". *IEEE Control Systems Magazine*, pages 6–17, April 1990.

[138] A. Michel, J. A. Farrell, and W. Porod. "Qualitative Analysis of Neural Networks". *IEEE Transactions on Circuits and Systems*, CAS–36(2):229–243, February 1989.

[139] A. N. Michel, S. Jie., and Y. Gune. "Analysis and Synthesis of a Class of Discrete-Time Neural Network Described on Hybercubes". *IEEE Trans. Neural Networks*, NN–2:32–46, 1991.

[140] M. Minskey and S. Papert. *Perceptrons*. MIT Press, Cambridge, MA, 1969.

[141] G. Mirchandani and W. Cao. "On Hidden Nodes for Neural Nets". *IEEE Transactions on Circuits and Systems*, CAS–36(5):661–664, May 1989.

[142] Y. Miyazaki, N. Goto, and Y. Kanayama. "Optical Signal Processing Using Waveguide Type Accost-Optic Switch". In *1988 ULTRASONICS Symp. w-4.*, pages 447–450, Chicago. USA, 1988.

[143] M. L. Mnmford, D. k. Andes, and L. R. Kern. "The Mod 2 Neurocomputer System Design". *IEEE Trans. Neural Networks*, NN–3(3):423–433, May 1992.

[144] B. L. Montgomery and B. V. k. Vijaya Kumar. "Evaluation of the use of the Hopfield neural network model as a nearest-neighbor algorithm". *Applied Optics*, vol. 25(20):3759–3766, October 1986.

[145] B. Muller and J. Reinhardt. *Neural Networks: An Introduction*. Springer-Verlag Berlin Heidelberg, 1990.

[146] K. Murakami, H. Takechi, and T. Aibara. "On a Range of Desired Behavior and Estimating the Ability of Associative Memory". *Trans. IECE*, J59–D(12):929–936, December 1976.

[147] K. S. Narendra and S. Mukhopadhyay. "Associative Learning in Random Environments Using Neural Networks". *IEEE Trans. Neural Networks*, NN–2(1):20–31, January 1991.



[148] N. M. Nasrabadi and C. Y. Choo. "Hopfield Network for Stereo Vision Correspondence". *IEEE Trans. Neural Networks*, NN–3(1):5–13, January 1992.

[149] C. Neti, M. H. Schneider, and E. D. Young. "Maximally Fault Tolerant Neural Networks". *IEEE Trans. Neural Networks*, 3(1):14–23, January 1992.

[150] D. H. Nguyen and B. Widrow. "Neural Networks for Self-Learning Control Systems". *IEEE Control Systems Magazine*, pages 18–23, April 1990.

[151] H. Nishimori. "On the Dynamics of Neural Network". *Mathematical Sciences*, pages 10–13, August 1991.

[152] Y. Owechko. "Nonlinear Holographic Associative Memories". *IEEE Journal of Quantum Electronics*, vol. 25(3):619–634, March 1989.

[153] T. S. Parker and L. O. Chua. "Chaos: A Tutorial for Engineers". In *Proceedings of the IEEE*, volume 75, pages 982–1008, 1987.

[154] S. J. Perantonis and P. J. G. Lisboa. "Translation, Rotation, and Scale Invariant Pattern Recognition by High-Order Neural Networks and Moment Classifiers". *IEEE Trans. Neural Networks*, NN–3(2):241–251, March 1992.

[155] L. Personnaz, I. Guyon, and G. Dreyfus. "Collective computational properties of neural networks: New Learning mechanisms". *Physical Review A*, vol. 34(5):4217–4228, November 1986.

[156] M. M. Polycarpou and P. A. Ioannou. "Learning and Convergence Analysis of Neural-Type Structured Networks". *IEEE Trans. Neural Networks*, NN–3(1):39–50, January 1992.

[157] D. L. Prados and S. C. Kak. "Neural Network Capacity Using Delta rule". *Electronics Letters*, 25(3):197–199, February 1989.

[158] T. Proggio and F. Girosi. "Networks for Approximation and Learning". *Proceedings of the IEEE*, vol. 78(9):1481–1497, September 1990.

[159] D. Psaltis and N. Farhat. "Optical information processing based on an associative-memory model of neural nets with thresholding and feedback". *Optics Letters*, vol. 25(2):98–100, February 1985.

[160] D. Psaltis, K. Wagner, and D. Brady. "Learning in Optical Neural Computers". In *Proceedings of the 1st IEEE International Conference on Neural Networks (ICNN-87)*, volume II, page 495, 1987.

[161] G. N. Reeke, JR, O. Sporns, and G. M. Edelman. "Synthetic Neural Modeling: "Darwin" Series of Recognition Automata". *Proceedings of the IEEE*, vol. 78(9):1498–1530, September 1990.

[162] D. L. Reilly, C. Scofield, C. Elbaum, and L. N. Cooper. "Learning System Architectures Composed of Multiple Learning Modules". In *Proceedings of the 1st IEEE International Conference on Neural Networks (ICNN-87)*, volume II, pages 495–503, 1987.

[163] A. F. Rocha, I. R. Guilherme, M. Theoto, A. M. K. Miyadahira, and M. S. Koizumi. "A Neural Net for Extracting Knowledge from Natural Language Data Bases". *IEEE Trans. Neural Networks*, 3(5):819–828, September 1992.

[164] F. Rosenblatt. *Principles of Neurodynamics*. Spartan, New York, 1962.

[165] J. Rubner and K. Schulten. "Development of Feature Detectors by Self-Organization: A Network Model". *Biol. Cybern.*, 62:193–199, 1990.

[166] D. E. Rumelhart, G. E. Hinton, and R. J. Williams. "Learning representations by back-propagating errors". *Nature*, 323:533–536, 1986.



[167] D. E. Rumelhart, J. L. McClelland, et al. *Parallel Distributed Processing*. MIT, 1986.

[168] R. M. Sanner and J. E. Slotine. "Gaussian Networks for Direct Adaptive Control". *IEEE Trans. Neural Networks*, NN–3(6):837–863, November 1992.

[169] M. Sato. "A Learning Algorithm to Teach Spatiotemporal Patterns to Recurrent Neural Networks". *Biol. Cybern.*, 62:259–263, 1990.

[170] M. Sato. "A Real Time Learning Algorithm for Recurrent Analog Neural Networks". *Biol. Cybern.*, 62:237–241, 1990.

[171] T. Sejnowski and C. Rosenberg. "Parallel Networks that learn to pronounce English Text". *Complex Systems.*, vol. 58:145–168, 1987.

[172] J. J. Shynk. "Performance Surfaces of a Single-Layer Perceptron". *IEEE Trans. Neural Networks*, NN–1(3):268–274, September 1990.

[173] D. F. Specht. "A General Regression Neural Network". *IEEE Trans. Neural Networks*, NN–2(6):568–576, November 1991.

[174] M. Stenvenson, R. Winter, and B. Widrow. "Sensitivity of Feedforward Neural Networks to Weight Errors". *IEEE Trans. Neural Networks*, NN–1(1):71–80, March 1990.

[175] H. M. Stoll and L. S. Lee. "Continuous Time Optical Neural Network". In *Proceedings of the 2nd IEEE International Conference on Neural Networks (ICNN-88)*, page 495, 1988.

[176] D. G. Stork. "Self-Organization, Pattern Recognition, and Adaptive Resonance Networks". *Journal of Neural Network Computing*, vol. 1(1):26–42.

[177] S. I. Sudharsanan and M. Sundareshan. "Equilibrium Characterization of Dynamical Neural Networks and a Systematic Synthesis Procedure for Associative Memories". *IEEE Trans. Neural Networks*, NN–2(5):509–521.

[178] H. M. Tai and T. L. Jong. "Neural Networks with Higher-Order Nonlinearity". *Electronics Letters*, 24(19):1225–1226, September 1988.

[179] Y. Takefuji, L.-L. Chen, K.-C. Lee, and J. Huffman. "Parallel Algorithm for Finding a Near-Maximum Independent Set of a Circle Graph". *IEEE Trans. Neural Networks*, NN–1(3):263–267, September 1990.

[180] D. W. Tank and J. J. Hopfield. "Simple 'Neural' Optimization Networks: An A/D Converter, Signal Decision Circuit, and a Linear Programming Circuit". *IEEE Transactions on Circuits and Systems*, CAS–33:533–541, May 1986.

[181] D. W. Tank and J. J. Hopfield. "Neural Computation by Concentrating Information in Time". In *Proceedings of the National Academy of Science. USA*, volume 84, pages 1896–1900, 1987.

[182] M. F. Tenorio and W.-T. Lee. "Self-Organizing Network for Optimum Supervised Learning". *IEEE Trans. Neural Networks*, NN–1(1):100–110, March 1990.

[183] R. F. Thompson. *The Brain.* W. H. Freeman, New York, 1985.

[184] R. F. Thompson. "The Neurobiology of Learning and Memory". *Science*, pages 941–947, August 1986.

[185] Ya. Z. Tspykin. *Foundations of the Theory of Learning Systems.* Academic Press, Orlando, FL, 1973.

[186] Y. Uesaka and K. Ozeki. "On Some Properties of Associative Memory". *Trans. IECE*, J55–D(5):323–330, May 1972.



[187] D. E. Van den Bout and T. K. Miller. "A Traveling Salesman Objective Function that Works ". In *Proceedings of the 2nd IEEE International Conference on Neural Networks (ICNN-88)*, volume 2, pages 299–304, 1988.

[188] D. E. Van den Bout and T. K. Miller. "Graph Partitioning Using Annealed Neural Networks". *IEEE Trans. Neural Networks*, NN–1(2):81–92, June 1990.

[189] A. T. Waibel, G. Hanazawa, S. K. Hinton, and K. Lang. "Phoneme Recognition Using Time-delay Neural Networks". *IEEE Transactions on Acoustics, Speech and Signal Processing*, March 1989.

[190] D. Wang and M. A. Arbib. "Complex Temporal Sequence Learning Based on Short-term Memory". *Proceedings of the IEEE*, vol. 78(9):1536–1543, September 1990.

[191] T. Wang, X. Zhuang, and X. Xing. "Weighted Learning of Bidirectional Associative Memories by Global Minimization". *IEEE Trans. Neural Networks*, 3(6):1010–1018, November 1992.

[192] Y. F. Wang, J. B. Cruz, and J. H. Mulligan. "Two Coding Strategies for Bidirectional Associative Memory". *IEEE Trans. Neural Networks*, NN–1:81–92, March 1990.

[193] Y. F. Wang, J. B. Cruz, and J. H. Mulligan. "Guaranteed Recall of All Training Pairs for Bidirectional Associative Memory". *IEEE Trans. Neural Networks*, NN–2(6):559–567, November 1991.

[194] W. G. Wee. "Generalized Inverse Approach to Clustering, Feature Detection and Classification". *IEEE Transactions on Information Theory*, IT–17:362–369, May 1971.

[195] G. Weisbuch. "Scaling Laws for the Attractors of Hopfield Networks". *Journal de Physique Letters*, 46:623–630, 1985.

[196] P. Werbos. *Beyond Regression: New Tools for Prediction and Analysis in the Behavioral Sciences.* Ph.D. thesis, Harvard University, 1974.

[197] H. White. "Learning in Artificial Neural Networks: A Statistical Perspective". *Neural Computation*, 1:425–469, 1989.

[198] H. White. "Neural Network Learning and Statistics". *AI Expert*, pages 48–52, December 1989.

[199] B. Widrow and M. A. Lehr. "30 Years of Adaptive Neural Networks: Perceptron, Madaline, and Back-propagation". *Proceedings of the IEEE*, vol. 78(9):1415–1442, September 1990.

[200] B. Widrow and S. D. Stearns. *Adaptive Signal Processing.* Prentice Hall, Englewood Cliffs, NJ, 1985.

[201] B. Widrow and R. Winter. "Neural Nets for Adaptive Filtering and Adaptive Pattern Recognition". *IEEE Computer Magazine*, pages 25–39, March 1988.

[202] R. J. Williams and D. Zipser. "A Learning Algorithm for Continually Running Fully Recurrent Neural Networks". *Neural Computation*, 1:270–280, 1989.

[203] G. V. Wilson and G. S. Pawley. "On the Stability of the Travelling Salesman Problem Algorithm of Hopfield and Tank". *Biological Cybernetics*, vol. 58:63–70, 1988.

[204] S. Yoshizawa, M. Morita, and S. Amari. "Capacity of Autocorrelation Type Associative Memory with Non-Monotonic Output Characteristics". *Technical report of IEICE*, NC 91–90, 1991.



[205] W. Zhang and S. Chen. "A Logical Architecture for Cognitive Maps". In *Proceedings of the 2nd IEEE International Conference on Neural Networks (ICNN-88)*, volume 1, pages 231–238, 1988.

[206] X. Zhuang, Y. Huang, and S Chen. "Better Learning For Bidirectional Associative Memory". *Neural networks*, 6:1131–1146, 1993.

## 研 究 発 表

### 論文


[1] X.Huang, Y.Miyazaki: "Connected Associative Memory Neural Network with Dynamical Threshold Function", IEICE Trans. Inf. & Syst., Vol.E75-D, No.1, pp.170-179 (1992).

[2] 黄、宮崎:「半直交連想記憶ニューラルネットワーク」, 電子情報通信学会論文誌, Vol.J77-D-II, No.1, pp.228-235 (1994).

[3] X.Huang, Y.Miyazaki: "Capacity of Semi-Orthogonally Associative Memories Model", (Submitted to IEICE Trans. Inf. & Syst.,)


### 国際会議


[1] X.Huang, Y.Miyazaki: "The Capacity of the Semi-Orthogonally Associative Memories Model" Proc. Int. Joint Conference on Neural Networks (IJCNN'93), Nagoya, MemF-16, pp.2666-2669 (1993).


### 研究会


[1] X.Huang, Y.Miyazaki: "Connected Associative Memory Neural Network with Dynamical Threshold Function", 電子情報通信学会信学技報, Vol.90, No.428, NC90-56 (1991).

[2] 黄、宮崎:「相互結合型ニューラルネットワークの一般化と半直交学習法」, 重点領域研究「超高速・超並列光エレクトロニクス」第3回研究会, pp.75-78 (1991).

[3] 黄、宮崎:「半直交連想記憶モデルの理論解析」, 電子情報通信学会信学技報, NC92-82, pp.15-22 (1992).

全国大会及び東海支部大会

[1] 黄、宮崎:「ホップフィルトニューラルネットワークの二次元化モデルと安定性に関して検討」, 電子情報通信学会秋季全国大会, SD-11-7 (1989).

[2] 黄、宮崎:「Hopfield neural network model の二次元モデル化と安定性の検討」, 電気関係学会東海支部連合大会, pp.533 (1989).

[3] 黄、宮崎:「多層Hopfieldニューラルネットワークモデル」, 電子情報通信学会春季全国大会, D-37 (1990).

[4] 黄、宮崎:「相互結合型ニューラルネットワークの収束性と半直交モデル」, 電子情報通信学会秋季全国大会, D-25 (1990).

[5] 黄、宮崎:「半直交相互結合型ニューラルネットワークモデル」, 電気関係学会東海支部連合大会, pp.5 (1990).

[6] 黄、宮崎:「動的なしきい関数を用いた相互結合型ニューラルネットワークモデル」, 電子情報通信学会春季全国大会, SD-2-6 (1991).

[7] 黄、宮崎:「無条件総合的な安定性を持つ相互結合型ニューラルネットワーク」, 電子情報通信学会秋季全国大会, SD-2-1 (1991).

[8] 黄、宮崎:「相互結合型ニューラルネットワークの総合安定条件とその特徴」, 電気関係学会東海支部連合大会, pp.8 (1991).

[9] 黄、宮崎:「半直交学習法を用いた相互結合型ニューラルネットワークの動的な特性」, 電子情報通信学会春季全国大会, SD-2-2 (1992).



[10] 黄、宮崎:「半直交連想記憶モデルの記憶能力」, 電子情報通信学会秋季全国大会, SD-2-1, pp.(6-375)-(6-376) (1992).

[11] 黄、宮崎:「SAMニューラルネットワークモデルの記憶能力」, 電気関係学会東海支部連合大会, 579, pp.290 (1992).

[12] 黄、宮崎:「直交連想を有する相互結合型ニューラルネットワークモデル」, 電子情報通信学会春季全国大会, SD-2-1, pp.(6-253)-(6-254) (1993).

[13] 黄、宮崎:「半直交連想記憶モデルにおける最適な特徴パラメータ」, 電子情報通信学会秋季全国大会, D-13, pp.(6-13)-(6-14) (1993).

[14] 黄、宮崎:「半直交連想記憶モデルにおける特徴パラメータの特性」, 電気関係学会東海支部連合大会, 582, pp.291 (1993).

# ACKNOWLEDGMENTS

This thesis was accomplished while I was studying as a doctoral candidate at Department of Information and Computer Science, Toyohashi University of Technology (TUT), Japan, since April, 1989 to January, 1994. I am greatly indebted to a large number of people for making this research possible.

Firstly, I wish to express my sincere gratitude to Professor Yasumitsu Miyazaki (TUT, Japan) for his kindling my interest to this exciting area and providing me a free academic environment to carry out the research. More importantly, his inspiring guidance and insisting encouragement, not only on the research but also on many other aspects, have contributed a lot to the progress of the research.

I would be very grateful to Professors Yoshifusa Ito, Shiro Usui, and Seiichi Nakagawa (TUT, Japan) for their helpful comments and suggestions in the examination of this thesis. I also wish to express my appreciations to Professors Zeng-Rong Jiang, Yu Sha, Ji-Chang Sha and Qing-Hua Chen (Changsha University of Technology, China) for their kind guidance and constant encouragement, and to Professors Yoshiaki Tadokoro (TUT, Japan), Hidemitsu Ogawa (Tokyo University of Technology, Japan) and Dr. Kenichi Funahashi (TUT, Japan) for their helpful comments and discussions at the institutes.

Many thanks are also due to Associate Professor Nobuo Goto (TUT, Japan) for his suggestions and discussions. I would like to thank Mr. Haruo Miyawaki, the technician from the Information Transmission Laboratory of

TUT, Japan for his kind assistance. I also would like thank Dr. Talabattula Srinivas, Mr. Junichi Minamoto, and all of the seniors and juniors from the Information Transmission Laboratory of TUT, Japan for their hospitality and helping hands.

Special thanks are owing to State Education Commission of the People's Republic of China, and Changsha University of Technology, China for affording to me the opportunity of studying in Japan and financial support, and to the Ministry of Education (Monbusho) of Japan for the financial support and the agreeable accommodations.

I would acknowledge many of my Japanese and Chinese friends whose kindness, friendship, hospitality made my stay along with my wife and son in Toyohashi so enjoyable and unforgettable. I am sorry I can not name them because of too many.

Last, but never least, I would like to give my deep gratitude to my wife and son, my parents and all other member of my family, who are always a great source of support and encouragement during the research that took so much time of me away from them.

Xin Min HUANG.
Toyohashi, Japan.